M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

SOLUTIONS TO ASSESSED COURSEWORK 1

Exercise 5.6. Let (X, d) be the metric space in question. If $y \in B_{\epsilon/2}(x)$ then $d(x, y) < \epsilon/2$. For all $z \in B_{\epsilon/2}(y)$, we have $d(y, z) < \epsilon/2$ and thus $d(x, z) < d(x, y) + d(y, z) < \epsilon/2 + \epsilon/2 = \epsilon$. (Here we used the triangle inequality.) Thus $z \in B_\epsilon(x)$. Since z was an arbitrary point of $B_{\epsilon/2}(y)$, it follows that $B_{\epsilon/2}(y) \subseteq B_\epsilon(x)$.

Exercise 6.8. From lectures or reading (Proposition 6.18 on p65 of the textbook) we have that, for a subset A of a metric space (X, d), the closure $\overline{A}$ is equal to the union of A with the limit points of A. Thus we need to show that the limit points of $B = B_1(0, 0)$ are precisely

$$L = \{(x, y) \in \mathbb{R}^2 : d\big((x, y), (0, 0)\big) = 1\}.$$

Write $O = (0, 0) \in \mathbb{R}^2$. Suppose that $z \in \mathbb{R}^2$ is a limit point of B. Then certainly $d(z, O) \geq 1$, for if $d(z, O) < 1$ then $z \in B$ and so z is not a limit point of B. Suppose now that $d(z, O) > 1$. Set $\epsilon = \frac{1}{2}(d(O, z) - 1)$. For $y \in B_\epsilon(z)$ we have $d(O, z) < d(O, y) + d(y, z) < d(O, y) + \frac{1}{2}(d(O, z) - 1)$, hence $d(O, y) > 1 + \frac{1}{2}(d(O, z) - 1) > 1$. In other words, $y \notin B$. So if $d(z, O) > 1$, the ball of radius ϵ about z contains no point of B, and so z is not a limit point of B. We have shown that if z is a limit point of B, then $z \in L$. On the other hand, any point of L is obviously a limit point of B – for instance, $x \in L$ is the limit of the sequence $x_n = (1 - 1/n)x$ with $x_n \in B$ – so the set of limit points of B is precisely L, as required.

Exercise 6.25. Let $(X, d_X$ and (Y, d_Y) be metric spaces. We need to show that a map $f\colon X \to Y$ is continuous if and only if $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X. Suppose first that f is continuous, and that $x_n \to x$ in X. We need to show that $f(x_n) \to f(x)$; that is, we need to show that for all $\epsilon > 0$ there exists N such that $f(x_n) \in B_\epsilon(f(x))$ whenever $n \geq N$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at x, we have that there exists $\delta > 0$ such that $f(x') \in B_\epsilon(f(x))$ whenever $x' \in B_\delta(x)$. Since $x_n \to x$ we have that there exists N such that $x_n \in B_\delta(x)$ whenever $n \geq N$. So whenever $n \geq N$ we have that $f(x_n) \in B_\epsilon(f(x))$, as required.

For the converse, suppose that $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X. We need to show that f is continuous. Suppose not. Then there exists $x \in X$ such that f is not continuous at x; in other words, there exists $\epsilon > 0$ such that, for all $\delta > 0$ we can find $x' \in B_\delta(x)$ with $f(x') \notin B_\epsilon(f(x))$. Taking $\delta = 1/n$ we have that there exists $x_n \in X$ with $x_n \in B_{1/n}(x)$ but $f(x_n) \notin B_\epsilon(f(x))$. Thus $x_n \to x$, but $f(x_n) \not\to f(x)$, contradicting our original assumption. Thus f is continuous.

Exercise 7.6. We check the axioms for a topological space. (T1) is obvious. For (T2), we need to show that given U, $V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$. This is immediate if either U or V is either $\varnothing$ or $\mathbb{R}$; otherwise $U = (-\infty, a)$, $V = (-\infty, b)$ for some a, $b \in \mathbb{R}$, and

$$U \cap V = \big(-\infty, \min(a, b)\big) \in \mathcal{T}.$$

For (T3) let U_i, $i \in I$, be elements of $\mathcal{T}$. We need to show that $\bigcup_{i \in I} U_i \in \mathcal{T}$. This is obvious if $U_i = \mathbb{R}$ for any i, and without loss of generality we may assume that $U_i \neq \varnothing$ for all $i \in I$. Then $U_i = (-\infty, b_i)$ for some $b_i \in \mathbb{R}$. Either

$$B = \{b_i : i \in I\}$$

is bounded above or it is not. If B is bounded above then $\bigcup_{i \in I} U_i = (-\infty, b)$ with $b = \sup B$; otherwise $\bigcup_{i \in I} U_i = \mathbb{R}$. In either case, $\bigcup_{i \in I} U_i \in \mathcal{T}$.

Exercise 9.5.

(a) The statement is false. Take $X = Y = \mathbb{R}$ with the Euclidean topology, $f(x) = e^x$, and $A = \mathbb{R}$. Then A is closed in X, but $f(A) = (0, \infty)$ is not closed in Y.

(b) The statement is false. Take $X = \mathbb{R}$ with the Euclidean topology, $A = (0, 1)$, and $B = X$. Then $A \cap \overline{B} = (0, 1)$, but $A \cap B = A$ and thus $\overline{A \cap B} = [0, 1]$.

(c) The statement is false. Let S be any set with $|S| > 1$. Let X be S equipped with the discrete topology, and let Y be S equipped with the indiscrete topology. Let $f\colon X \to Y$ be the identity map; this is continuous because any map out of a topological space with the discrete topology is continuous. (We proved this in lectures.) Let $B = \{s\}$, where $s \in S$, and regard B as a subset of the topological space Y. Then $\overline{B} = Y$, as Y has the indiscrete topology, so $f^{-1}(\overline{B}) = X$. And $\overline{f^{-1}(B)} = f^{-1}(B) = \{s\}$, as X has the discrete topology, so $\overline{f^{-1}(B)} \neq f^{-1}(\overline{B})$.

Exercise 9.14.

(a) For any subset B of A, we have that $C \subseteq A \iff C \setminus B \subseteq A$. Applying this with $C = \overline{A}$ and $B = \mathring{A}$ we have that

$$\partial A \subseteq A \iff \overline{A} \setminus \mathring{A} \subseteq A \iff \overline{A} \subseteq A \iff A \text{ is closed.}$$

(b) Suppose that $\partial A = \varnothing$. Then $\overline{A} = \mathring{A}$, and since $\mathring{A} \subseteq A \subseteq \overline{A}$ it follows that $\mathring{A} = A$ and $A = \overline{A}$. Thus A is both open and closed. Conversely, suppose that A is both open and closed. Then $\mathring{A} = A = \overline{A}$, and so $\partial A = \overline{A} \setminus \mathring{A} = A \setminus A = \varnothing$.

Exercise 8.7 (which was optional and worth no extra marks). It suffices to show that, for every open subset U of $\mathbb{R}^2$ and every $(z, w) \in U$, there exist rational numbers x, y and a positive rational number q such that $(z, w) \in B_q(x, y)$ and $B_q(x, y) \subseteq U$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(z, w) \subseteq U$. Choose rational numbers x and y such that $|z - x| < \frac{\epsilon}{3\sqrt{2}}$ and $|w - y| < \frac{\epsilon}{3\sqrt{2}}$. Then

$$d\big((z, w), (x, y)\big) = \sqrt{|z - x|^2 + |w - y|^2} < \sqrt{\frac{\epsilon^2}{18} + \frac{\epsilon^2}{18}} = \frac{\epsilon}{3}$$

Choose a rational number q such that $\frac{\epsilon}{3} < q < \frac{\epsilon}{2}$. Then $(z, w) \in B_{\epsilon/3}(x, y) \subset B_q(x, y)$, and for all $(x', y') \in B_q(x, y)$ we have that

$$d\big((z, w), (x', y')\big) \leq d\big((z, w), (x, y)\big) + d\big((x, y), (x', y')\big) < \frac{\epsilon}{3} + q < \frac{\epsilon}{3} + \frac{\epsilon}{2} < \epsilon.$$

So $B_q(x, y) \subset B_\epsilon(z, w)$ and therefore $B_q(x, y) \subset U$, as required. Essentially the same argument proves that

$$\Big\{ B_q(x_1, \ldots, x_n) : q > 0 \text{ rational; } x_1, \ldots, x_n \text{ rational} \Big\}$$

is a basis for the Euclidean topology on $\mathbb{R}^N$.