Solutions to Assessed Coursework 2

Exercise 12.11. Recall that a topological space X is connected if the following condition is satisfied: whenever $X = C \cup D$ with C and D closed subsets of X such that $C \cap D = \emptyset$, then either C or D is empty.

(a) False. A counterexample is the following: $X = Y = [0, 3]$, $A = B = [1, 2]$.

(b) False. A counterexample is the following: $X = \mathbb{R}$, $A = (0, 1]$, $B = (-\infty, 0] \cup [1, +\infty)$.

(c) True. Let A, B be two closed subsets of the topological space X such that both $A \cap B$ and $A \cup B$ are connected. We want to show that A and B are connected. By symmetry it is enough to show that A is connected. Let's assume that $A = C \cup D$ where C and D are open and closed in A and such that $C \cap D = \emptyset$. We need to show that either C is empty or D is empty. It is clear that $A \cap B = (C \cap B) \cup (D \cap B)$ and the union is disjoint. Notice that every set we are considering is a closed subset of X. Since $A \cap B$ is connected, either $C \cap B$ or $D \cap B$ is empty. So we distinguish two cases.

1. $C \cap B = \emptyset$**:** now we see that $A \cup B = (C \cup D) \cup B = C \cup (D \cup B)$ and that $C \cap (D \cup B) = \emptyset$. We have decomposed $A \cup B$ into the disjoint union of two closed subsets. Since $A \cup B$ is connected, either C or $D \cup B$ is empty. In particular, either C or D is empty.

2. $D \cap B = \emptyset$**:** this case is completely symmetrical. Use $A \cup B = D \cup (C \cup B)$.

Exercise 13.4 Recall that for the subsets $X \subseteq \mathbb{R}^n$ the following characterisation holds

$$X \subseteq \mathbb{R}^n \text{ is compact} \iff X \text{ is closed and bounded} \tag{1}$$

Recall that for a metric space (X, d_X) the following characterisation holds

$$(X, d_X) \text{ is compact} \iff X \text{ is sequentially compact} \tag{2}$$

(1) $[0, 1) \subseteq \mathbb{R}$ is not compact because it is not closed. Alternatively, notice that $\{(-1, 1 - 1/n)\}_{n>0}$ is an open cover of $[0, 1)$ which does not admit a finite sub cover.

(2) $[0, +\infty) \subseteq \mathbb{R}$ is not compact because it is not bounded. Alternatively, notice that $\{(-1, n)\}_{n>0}$ is an open cover of $[0, +\infty)$ which does not admit a finite sub cover.

(3) $\mathbb{Q} \cap [0, 1]$ is a metric space with the standard euclidean metric induced from $\mathbb{R}$. It is not compact as the sequence $\{x_n := 1/3(1 + 1/n)^n\}_{n>0}$ is in $\mathbb{Q} \cap [0, 1]$ for any n, but the limit $e/3$ is irrational.

(4) $X := \{(x, y) \in \mathbb{R}^2 \,;\, x^2 + y^2 = 1\} \subset \mathbb{R}^2$ is compact as it is closed and bounded:
- X is bounded as it is contained in any ball of radius $r > 1$, for example $X \subset B_2((0, 0))$
- X is closed as it is the pre-image of a closed set under the continuous function $f \colon \mathbb{R}^2 \to \mathbb{R}$, both with the topology induced by the euclidean metric, defined by $f(x, y) = x^2 + y^2$. By definition $X = f^{-1}(\{1\})$ and $\{1\}$ it is obviously closed in $(\mathbb{R}, d_2)$.

(5) $X := \{(x, y) \in \mathbb{R}^2 \,;\, |x| + |y| \leq 1\} \subset \mathbb{R}^2$ is compact as it is closed and bounded:
- X is bounded as it is contained in any ball of radius $r > 1$, for example $X \subset B_2((0, 0))$

- X is closed as it is the pre-image of a closed under the continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$, both with the topology induced by the euclidean metric, defined by $f(x,y) = |x| + |y|$. By definition $X = f^{-1}((-\infty, 1])$ and $(-\infty, 1]$ it is obviously closed in $(\mathbb{R}, d_2)$.

(6) $X := \{(x,y) \in \mathbb{R}^2\,;\, x^2 + y^2 < 1\} \subset \mathbb{R}^2$ is not compact as it is not closed. Alternatively, notice that $\{B_{r=1-1/n}((0,0))\}_{n>0}$ is an open cover of X which does not admit a finite sub cover.

(7) $X := \{(x,y) \in \mathbb{R}^2\,;\, x \geq 1, 0 \leq 1 \leq 1/x\} \subset \mathbb{R}^2$ is not compact because it is not bounded. Alternatively, notice that $\{(0,n) \times \mathbb{R}\}_{n>0}$ is an open cover of X which does not admit a finite sub cover. In the previous cover the coordinate x belongs to the first set and the coordinate y to the second. The set is "unbounded in the x-direction"; for this reason we are not able to extract the sub cover.

Exercise 13.14 Let (X,d) be a compact metric space. Let $f\colon X \to X$ be a continuous map without fixed points. Consider the function $g\colon X \to \mathbb{R}$ defined by $g(x) = d(f(x), x)$ for each $x \in X$.

We need to show that g is continuous at any point $x_0 \in X$. Fix $\varepsilon > 0$. Since f is continuous at x_0, there exists $\delta > 0$ such that for any $x \in X$, $d(x, x_0) < \delta \Longrightarrow d(f(x), f(x_0)) < \frac{\varepsilon}{2}$. Now consider the inequalities

$$\begin{aligned} |g(x) - g(x_0)| &= |d(f(x), x) - d(f(x_0), x_0)| \\ &= |d(f(x), x) - d(f(x), x_0) + d(f(x), x_0) - d(f(x_0), x_0)| \\ &\leq |d(f(x), x) - d(f(x), x_0)| + |d(f(x), x_0) - d(f(x_0), x_0)| \\ &\leq d(x, x_0) + d(f(x), f(x_0)). \end{aligned}$$

This implies that, whenever $d(x,x_0) < \delta$ we have $|g(x) - g(x_0)| < d(x, x_0) + \frac{\varepsilon}{2}$. Now pick $\delta^* = \min\{\delta, \frac{\varepsilon}{2}\}$. We have that if $d(x, x_0) < \delta^*$ then $|g(x) - g(x_0)| < d(x, x_0) + \frac{\varepsilon}{2} < \delta^* + \frac{\varepsilon}{2} \leq \varepsilon$.

Since g is a real continuous function over the compact topological space X, it attains its minimum, i.e. there exists a point $\bar{x} \in X$ such that $g(\bar{x}) = \inf_X g$. Since $f(\bar{x}) \neq \bar{x}$, $\inf_X g = g(\bar{x}) = d(f(\bar{x}), \bar{x}) > 0$. And obviously $d(f(x), x) = g(x) \geq \inf_X g$ for any $x \in X$.

Exercise 14.11 Suppose the intersection is empty and let $U_n = X \setminus V_n$. Then U_n is open for each n and $U_j \subseteq U_{j+1}$. Moreover $\bigcup U_n = \bigcup (X \setminus V_n) = X \setminus \bigcap V_n = X$. Therefore by the compactness of X we only need finitely many of these, so $X = U_1 \cup \cdots \cup U_m$ for some m. Since $U_j \subseteq U_{j+1}$ it follows that $X = U_m$. However $U_m = X \setminus V_m$ which would imply V_m is empty, a contradiction.

Exercise 16.8 Let us recall the definition of uniformly continuous function and of uniformly convergent sequence of functions. We will give the definition and solve the exercise for real valued functions $f\colon D \to \mathbb{R}$, but everything we say makes sense for any continuous function between two metric spaces $f\colon (X, d_X) \to (Y, d_Y)$.

(a): $f\colon D \to \mathbb{R}$, is uniformly continuous if

$$\forall \epsilon > 0,\ \exists \delta > 0\ :\ \forall\, x, y \in D\ :\ d(x,y) < \delta \Longrightarrow |f(x) - f(y)| < \epsilon$$

Notice that the crucial difference with respect to the definition of continuous function is that here δ does not depend on the point $x \in D$ around which we are looking.

(b): A sequence $\{f_n\}$, $f_n\colon D \to \mathbb{R}$ is uniformly convergent to $f\colon D \to \mathbb{R}$ if

$$\forall \epsilon > 0,\ \exists N_\epsilon\ :\ \forall n \geq N_\epsilon \text{ and } \forall x \in D,\ |f_n(x) - f(x)| < \epsilon$$

Here the difference with the pointwise convergence is that here N_ϵ does not depend on x. Notice that this means that in particular $\sup_{x\in D} |f_n(x) - f(x)| < \epsilon$ for all $n \geq N_\varepsilon$, i.e. the uniform convergence is just the convergence in the sup norm.

Now that we recalled both the definition precisely the exercise is very easy. Let us fix any $\epsilon > 0$ then

(1) Since $f_n \to f$ uniformly, there exists N_ϵ such that

$$\forall n \geq N_\epsilon \text{ and } \forall x \in D,\ |f_n(x) - f(x)| < \epsilon/3$$

(2) Since any f_n in the sequence is uniformly continuous, there exists $\delta > 0$ such that

$$\forall\, x, y \in D\ :\ d(x, y) < \delta \Longrightarrow |f_{N_\epsilon}(x) - f_{N_\epsilon}(y)| < \epsilon/3$$

but then, for all $x, y \in D$ such that $d(x, y) < \delta$ we have

$$|f(x) - f(y)| = |f(x) - f_{N_\epsilon}(x) + f_{N_\epsilon}(x) - f_{N_\epsilon}(y) + f_{N_\epsilon}(y) - f(y)|$$
$$\leq |f(x) - f_{N_\epsilon}(x)| + |f_{N_\epsilon}(x) - f_{N_\epsilon}(y)| + |f_{N_\epsilon}(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3$$

where the first inequality is just the triangular inequality and the second follows from (1) and (2). By definition f is uniformly continuous.

Exercise 17.2. Let us start by proving the following lemma.

Lemma. *Let (X, d) be a complete metric space and let $Y \subseteq X$ be a subspace. Then: Y is a complete metric space with the induced distance if and only if Y is closed in X.*

Proof of the lemma. $\Longrightarrow$) Let us assume that Y is complete. We want to prove that it is closed. Let (y_n) be a sequence of elements of Y which converges to a point $x \in X$. We want to prove that $x \in Y$. It is easy to see that (y_n) is Cauchy in the metric space X. Therefore it is Cauchy also in the metric space Y. Since Y is complete, (y_n) converges to a point $y \in Y$. But the convergence holds also in X, so $x = y$ by the uniqueness of the limit.

$\Longleftarrow$) Let (y_n) be a Cauchy sequence in Y. It is obviously a Cauchy sequence in X. Since X is complete, (y_n) converges to $x \in X$. Since Y is closed in X, $x \in Y$. Therefore (y_n) is convergent in Y. □

Since $\mathbb{R}^n$ is complete, by the lemma, a subspace of $\mathbb{R}^n$ is complete if and only if it is closed in $\mathbb{R}^n$.

(i) $\{1/n \mid n \in \mathbb{N}^+\} \cup \{0\}$ is closed.

(ii) $\mathbb{Q} \cap [0, 1]$ is not closed: take your favourite sequence of rational numbers that converges to an irrational number, e.g.

$$\frac{1}{3}\left(1 + \frac{1}{n}\right)^n$$

as $n \in \mathbb{N}^+$. Indeed, this is a Cauchy sequence in $\mathbb{Q} \cap [0, 1]$ which does not converge in $\mathbb{Q} \cap [0, 1]$.

(iii) $X = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y \geq 1/x\} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ xy \geq 1\}$ is a closed subset of $\mathbb{R}^2$ because it is the preimage of the closed subset $[0, +\infty) \times [1, +\infty)$ of $\mathbb{R}^2$ under the continuous map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (x, xy)$.

Exercise 16.9. Let X be a compact topological space, let $f_n\colon X \to \mathbb{R}$, as $n \in \mathbb{N}$, be a sequence of continuous functions such that $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and

$x \in X$. Let us assume that the sequence (f_n) converges pointwise to the *continuous* function $f\colon X \to \mathbb{R}$. We want to show that the convergence is uniform, i.e.

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in X, |f_n(x) - f(x)| < \varepsilon, \tag{3}$$

Since our sequence is pointwise decreasing, we have that $f(x) = \inf_{n\in\mathbb{N}} f_n(x)$ for any $x \in X$. In particular the function $f_n - f$ is always non-negative. So the condition (3) is equivalent to

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in X, f_n(x) - f(x) < \varepsilon. \tag{4}$$

Now let's prove (4). Fix $\varepsilon > 0$. For any $n \in \mathbb{N}$, consider the set

$$U_{\varepsilon,n} = (f_n - f)^{-1}(-\infty, \varepsilon) = \{x \in X \mid f_n(x) - f(x) < \varepsilon\}.$$

Since f is continuous, also the difference $f_n - f$ is a continuous function, hence $U_{\varepsilon,n}$ is open as it is the preimage of the open interval $(-\infty, \varepsilon)$ under the continuous function $f_n - f$. Since our sequence is decreasing, it is easy to see that

$$U_{\varepsilon,0} \subseteq U_{\varepsilon,1} \subseteq U_{\varepsilon,2} \subseteq \cdots . \tag{5}$$

In other words we have an ascending sequence of open subsets of X.

Let's pick any point $x \in X$. Since the decreasing sequence of real numbers $(f_n(x))_n$ converges to $f(x)$, we have that there exists $\nu_{\varepsilon,x} \in \mathbb{N}$ such that

$$n \geq \nu_{\varepsilon,x} \Longrightarrow f_n(x) - f(x) < \varepsilon.$$

This implies that $x \in U_{\varepsilon,\nu_{\varepsilon,x}}$, i.e. x belongs to one (and actually infinitely many) member of our ascending sequence (5). Since x was any point of X, we have proved that

$$X = \bigcup_{n\in\mathbb{N}} U_{\varepsilon,n},$$

i.e. the open subsets in (5) constitute an open cover of X.

Since X is compact, it is possible to extract a finite subcover, i.e. $X = U_{\varepsilon,N_1} \cup \cdots \cup U_{\varepsilon,N_k}$. Set $N = \max\{N_1, \ldots, N_k\}$. Then it is clear that $X = U_{\varepsilon,N} = U_{\varepsilon,N+1} = \cdots$. So we have found $N \in \mathbb{N}$ such that, for any $n \geq N$ and any $x \in X$, $x \in U_{\varepsilon,n}$, i.e. $f_n(x) - f(x) < \varepsilon$. This is exactly (4).