M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

EXTRA PROBLEM SHEET

Exercise 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Define

$$\begin{aligned} d_p \colon\ (X \times Y) \times (X \times Y) &\longrightarrow \mathbb{R} \\ \big((x_1, y_1), (x_2, y_2)\big) &\longmapsto \sqrt[p]{d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p} \end{aligned}$$

A generalization of the Cauchy–Schwartz inequality, which you may assume without proof, is *Hölder's inequality*. This states that, if $p \in [1, \infty)$ and $q \in [1, \infty)$ are such that $1/p + 1/q = 1$, then:

$$\Big| \sum_{i=1}^{n} a_i b_i \Big| \leq \Big(\sum_{i=1}^{n} |a_i|^p \Big)^{1/p} \Big(\sum_{i=1}^{n} |b_i|^q \Big)^{1/q}$$

for all $a, b \in \mathbb{R}^n$. Use this to show that d_p is a metric on $X \times Y$.

Exercise 2. Let X denote the set of all continuous functions from the interval $[-1,1]$ to $\mathbb{R}$. Define:

$$d_1\colon X \times X \longrightarrow \mathbb{R} \qquad\qquad d_\infty\colon X \times X \longrightarrow \mathbb{R}$$

$$(f,g) \longmapsto \int_{-1}^{1} |(f(t) - g(t)|\, dt \qquad\qquad (f,g) \longmapsto \sup_{t\in[-1,1]} |f(t) - g(t)|$$

Both (X, d_1) and (X, d_∞) are metric spaces. (You do not need to prove this.) Consider the function

$$\mathrm{ev}_0\colon X \longrightarrow \mathbb{R}$$
$$f \longmapsto f(0)$$

Is ev_0 a continuous function on (X, d_1)? Is ev_0 a continuous function on (X, d_∞)?