M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

PROBLEM SHEET 1

Exercise 1. Let p be a prime number. Define a function $d\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ \frac{1}{r} & \text{if } m \neq n, \text{ where } m - n = p^{r-1}q \text{ with } q \in \mathbb{Z} \text{ not divisible by } p. \end{cases}$$

Show that d is a metric on $\mathbb{Z}$.

Exercise 2. Let $C([a,b])$ denote the set of continuous functions from $[a,b]$ to $\mathbb{R}$, and let $C^1([a,b])$ denote the set of differentiable functions $f\colon [a,b] \to \mathbb{R}$ such that f' is continuous. Let:

$$d_\infty(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

This defines a metric on both $C([a,b])$ and $C^1([a,b])$.

(1) Consider the map:

$$\textsc{Int} : \big(C([a,b]), d_\infty\big) \to \big(C^1([a,b]), d_\infty\big)$$
$$f \mapsto \int_a^x f(t)\,dt$$

Is $\textsc{Int}$ continuous?

(2) Consider the map:

$$\textsc{Diff} : \big(C^1([a,b]), d_\infty\big) \to \big(C([a,b]), d_\infty\big)$$
$$f \mapsto f'$$

Is $\textsc{Diff}$ continuous?

Exercise 3. Let (X, d) be a metric space and A be a subset of X. Show that $x \in \partial A$ if and only if, for all $\epsilon > 0$, we have that $B_\epsilon(x) \cap A$ and $B_\epsilon(x) \cap (X \setminus A)$ are both non-empty.

Exercise 4. Let (X, d) be a metric space and A be a subset of X. For $x \in X$, define

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

Show that:

(1) $d(x, A) = 0$ if and only if $x \in \overline{A}$.
(2) for all $y \in X$, $d(x, A) \leq d(x, y) + d(y, A)$.
(3) the map $x \mapsto d(x, A)$ defines a continuous function from X to $\mathbb{R}$.