M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

PROBLEM SHEET 2

Exercise 1

Let (X, d) be a metric space.

(1) Show that, given any two distinct points $x, y \in X$ there exist open sets U, V in X with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

(2) Suppose that B is a bounded subset of X and that $C \subset B$. Show that C is bounded, and that $\operatorname{diam} C \leq \operatorname{diam} B$.

Exercise 2

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by:

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0 \end{cases}$$

Show that the restriction of f to the subset $\mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$ is continuous. Is f continuous?

Exercise 3

(1) Let (X, d) be a metric space and let $A_1, \ldots, A_m$ be subsets of X. Show that:

$$\overline{\left(\bigcup_{i=1}^{i=m} A_i\right)} = \bigcup_{i=1}^{i=m} \overline{A_i}$$

(2) Let (X, d) be a metric space and let A_i, $i \in I$, be subsets of X. Show that:

$$\overline{\left(\bigcap_{i=1}^{i=m} A_i\right)} \subseteq \bigcap_{i=1}^{i=m} \overline{A_i}$$

(3) Give an example of a metric space (X, d) and subsets A, B of X such that:

$$\overline{A \cap B} \neq \overline{A} \cap \overline{B}$$

Exercise 4

Compute the closure of each of the following sets in $\mathbb{R}$:

(1) $[1, \infty)$
(2) $\mathbb{R} \setminus \mathbb{Q}$
(3) $\{\frac{n}{n+1} : n \in \mathbb{N}\}$
(4) $\{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\} \cup \{0, 1, 2\}$

Exercise 5

The *Cantor set* C is defined as follows. Let $C_0 = [0, 1]$ and, for $n \geq 0$, let C_{n+1} be the set obtained from C_n by taking each maximal closed interval I contained in C_n and removing from I the open interval that forms the middle third of I. Set:

$$C = \bigcap_{n \geq 0} C_n$$

(1) Show that C_n is a disjoint union of 2^n closed intervals;
(2) Show that C is closed.
(3) Show that C is non-empty.
(4) Optional: show that C is uncountable.
(5) Optional: show that every point in C is an accumulation point.
(6) Optional: show that C has empty interior.

The Cantor set is an example of an uncountable set with Lebesgue measure zero. Items (2) and (5) here show that C is a so-called *perfect set*. Item (6) shows that C is *nowhere dense* in $\mathbb{R}$.