M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

PROBLEM SHEET 5

Exercise 1. Let $(X, \mathcal{T})$ be a topological space. Let ∞ be an object not in X, and set:

$$\check{X} = X \cup \{\infty\}$$
$$\check{\mathcal{T}} = \mathcal{T} \cup \big\{V \cup \{\infty\} : V \subseteq X \text{ such that } X \setminus V \text{ is compact and closed}\big\}$$

(1) Show that $\check{\mathcal{T}}$ is a topology on $\check{X}$.
(2) Show that the topological space $(\check{X}, \check{\mathcal{T}})$ is compact.
(3) Show that the topological space $(\check{X}, \check{\mathcal{T}})$ contains $(X, \mathcal{T})$ as a subspace.
(4) Suppose that $X = \mathbb{R}$ with the usual topology. What is $(\check{X}, \check{\mathcal{T}})$?
(5) Suppose that $X = \mathbb{R}^2$ with the usual topology. What is $(\check{X}, \check{\mathcal{T}})$?

The space $(\check{X}, \check{\mathcal{T}})$ is called the *one-point compactification* of X.

Exercise 2.

(1) Show that the annulus $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$ is path connected.
(2) Show that the region $\{(x, y) \in \mathbb{R}^2 : x < 0 \text{ or } x > 1\}$ is not path-connected.

Exercise 3. Consider the *topologist's sine curve* T, which is the subspace of $\mathbb{R}^2$ defined by:

$$T = \{(0,0)\} \cup \left\{(x, \sin x^{-1}) : x \in (0,\infty)\right\}$$

(1) Draw a picture of T.
(2) Show that T is connected.
(3) Show that T is not path-connected.