M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

PROBLEM SHEET 8

Exercise 1. Consider the following functions $f_n : [0,1] \to \mathbb{R}$. In which case does the sequence $(f_n)_{n\geq 1}$ converge uniformly on $[0,1]$?

(1) $f_n(x) = \frac{x}{1+nx}$
(2) $f_n(x) = \frac{x^n}{1+x^n}$
(3) $f_n(x) = nx(1-x^2)^{n^2}$

Exercise 2. Construct functions $f_n : \mathbb{R} \to \mathbb{R}$ such that none of the f_n is continuous at $0 \in \mathbb{R}$ but that $(f_n)_{n\geq 1}$ converges uniformly on $\mathbb{R}$ to a continuous function.

Exercise 3. Let X be the metric space $[1, \infty)$, considered as a subspace of $\mathbb{R}$. Let $f : X \to X$ be the map $x \mapsto x + x^{-1}$. Show that:

(1) X is complete;
(2) $|f(x) - f(y)| < |x - y|$ for all $x, y \in X$;
(3) f has no fixed point.

Exercise 4. Recall the definition of homotopy from Problem Sheet 6. We proved there that setting $\gamma_1 \sim \gamma_2$ if and only if γ_1 is homotopic to γ_2 defines an equivalence relation on paths. Let X be a topological space, and choose a point $x_0 \in X$. The set

$$\pi_1(X, x_0) = \big\{[\gamma] : \gamma \text{ is a loop in } X \text{ based at } x_0\big\}$$

is called the *fundamental group* of X with basepoint x_0. The group operation $*$ is defined by setting the product of loops γ_1 and γ_2 to be the loop $\gamma_1 * \gamma_2$ given by

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that:

- setting $[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2]$ gives a well-defined operation $*$ on $\pi_1(X, x_0)$.
- the constant loop e_{x_0} given by $e_{x_0}(t) = x_0$ is an identity element for $*$.
- if we define the loop γ^{-1} by $\gamma^{-1}(t) = \gamma(1-t)$ then $[\gamma]*[\gamma^{-1}] = [e_{x_0}]$ and $[\gamma^{-1}]*[\gamma] = [e_{x_0}]$.
- the product $*$ on $\pi_1(X, x_0)$ is associative.

This shows that the fundamental group is, in fact, a group.

Exercise 5. Let X and Y be topological spaces, let x_0 be a point of X, let $f\colon X \to Y$ be a continuous map, and let $y_0 = f(x_0)$. Show that setting $f_\star[\gamma] = [f \circ \gamma]$ gives a well-defined group homomorphism $f_\star\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.