M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

EXTRA PROBLEM SHEET

Exercise 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Define

$$\begin{aligned} d_p\colon\ (X \times Y) \times (X \times Y) &\longrightarrow \mathbb{R} \\ \big((x_1, y_1), (x_2, y_2)\big) &\longmapsto \sqrt[p]{d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p} \end{aligned}$$

A generalization of the Cauchy–Schwartz inequality, which you may assume without proof, is *Hölder's inequality*. This states that, if $p \in [1, \infty)$ and $q \in [1, \infty)$ are such that $1/p + 1/q = 1$, then:

$$\Big|\sum_{i=1}^{n} a_i b_i\Big| \le \Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p} \Big(\sum_{i=1}^{n} |b_i|^q\Big)^{1/q}$$

for all $a, b \in \mathbb{R}^n$. Use this to show that d_p is a metric on $X \times Y$.

Solution. (M1) and (M2) are straightforward and I will omit them. For (M3), let x_1, x_2, $x_3 \in X$ and y_1, y_2, $y_3 \in Y$ be arbitrary. We need to prove that:

$$\sqrt[p]{d_X(x_1, x_3)^p + d_Y(y_1, y_3)^p} \le \sqrt[p]{d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p} + \sqrt[p]{d_X(x_2, x_3)^p + d_Y(y_2, y_3)^p}$$

Since everything is real and non-negative, this is equivalent to:

$$d_X(x_1, x_3)^p + d_Y(y_1, y_3)^p \le \Big(\sqrt[p]{d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p} + \sqrt[p]{d_X(x_2, x_3)^p + d_Y(y_2, y_3)^p}\Big)^p$$

But

$$\begin{aligned} d_X(x_1, x_3)^p + d_Y(y_1, y_3)^p &\le (d_X(x_1, x_2) + d_X(x_2, x_3))^p + (d_Y(y_1, y_2) + d_Y(y_2, y_3))^p \\ &= d_X(x_1, x_2)(d_X(x_1, x_2) + d_X(x_2, x_3))^{p-1} + \\ &\quad\ d_X(x_2, x_3)(d_X(x_1, x_2) + d_X(x_2, x_3))^{p-1} + \\ &\quad\ d_Y(y_1, y_2)(d_Y(y_1, y_2) + d_Y(y_2, y_3))^{p-1} + \\ &\quad\ d_Y(y_2, y_3)(d_Y(y_1, y_2) + d_Y(y_2, y_3))^{p-1} \end{aligned}$$

Consider the first and third terms here, and apply Hölder's inequality (with $q = p/(p-1)$, so that $1/p + 1/q = 1$):

$$d_X(x_1, x_2)(d_X(x_1, x_2) + d_X(x_2, x_3))^{p-1} + d_Y(y_1, y_2)(d_Y(y_1, y_2) + d_Y(y_2, y_3))^{p-1} \le$$

$$\Big(d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p\Big)^{1/p} \Big((d_X(x_1, x_2) + d_X(x_2, x_3))^{(p-1)q} + (d_Y(y_1, y_2) + d_Y(y_2, y_3))^{(p-1)q}\Big)^{1/q}$$

Do the same thing with the second and fourth terms above, and use $(p-1)q = p$ and $1/q = 1 - 1/p$, to find:

$$\begin{aligned} &(d_X(x_1, x_2) + d_X(x_2, x_3))^p + (d_Y(y_1, y_2) + d_Y(y_2, y_3))^p \le \\ &\quad \Big(d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p\Big)^{1/p} \Big((d_X(x_1, x_2) + d_X(x_2, x_3))^p + (d_Y(y_1, y_2) + d_Y(y_2, y_3))^p\Big)^{1-1/p} \\ &+ \Big(d_X(x_2, x_3)^p + d_Y(y_2, y_3)^p\Big)^{1/p} \Big((d_X(x_1, x_2) + d_X(x_2, x_3))^p + (d_Y(y_1, y_2) + d_Y(y_2, y_3))^p\Big)^{1-1/p} \end{aligned}$$

Cancelling common factors and rearranging gives:

$$\Big((d_X(x_1,x_2)+d_X(x_2,x_3))^p+(d_Y(y_1,y_2)+d_Y(y_2,y_3))^p\Big)^{1/p} \leq \big(d_X(x_1,x_2)^p+d_Y(y_1,y_2)^p\big)^{1/p}+\big(d_X(x_2,x_3)^p+d_Y(y_2,y_3)^p\big)^{1/p}$$

Or in other words:

$$(d_X(x_1,x_2)+d_X(x_2,x_3))^p+(d_Y(y_1,y_2)+d_Y(y_2,y_3))^p \leq \Big(\big(d_X(x_1,x_2)^p+d_Y(y_1,y_2)^p\big)^{1/p}+\big(d_X(x_2,x_3)^p+d_Y(y_2,y_3)^p\big)^{1/p}\Big)^p$$

So:

$$\begin{aligned} d_X(x_1,x_3)^p+d_Y(y_1,y_3)^p &\leq (d_X(x_1,x_2)+d_X(x_2,x_3))^p+(d_Y(y_1,y_2)+d_Y(y_2,y_3))^p \\ &\leq \Big(\sqrt[p]{d_X(x_1,x_2)^p+d_Y(y_1,y_2)^p}+\sqrt[p]{d_X(x_2,x_3)^p+d_Y(y_2,y_3)^p}\Big)^p \end{aligned}$$

as required. Thus (M3) holds. □

Exercise 2. Let X denote the set of all continuous functions from the interval $[-1,1]$ to $\mathbb{R}$. Define:

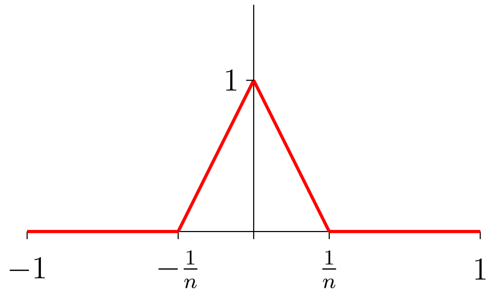
$$\begin{aligned} d_1\colon X\times X &\longrightarrow \mathbb{R} \\ (f,g) &\longmapsto \int_{-1}^{1} |(f(t)-g(t)|\,dt \end{aligned} \qquad \begin{aligned} d_\infty\colon X\times X &\longrightarrow \mathbb{R} \\ (f,g) &\longmapsto \sup_{t\in[-1,1]} |f(t)-g(t)| \end{aligned}$$

Both (X,d_1) and (X,d_∞) are metric spaces. (You do not need to prove this.) Consider the function

$$\begin{aligned} \mathrm{ev}_0\colon X &\longrightarrow \mathbb{R} \\ f &\longmapsto f(0) \end{aligned}$$

Is ev_0 a continuous function on (X,d_1)? Is ev_0 a continuous function on (X,d_∞)?

Solution. ev_0 *is not* a continuous function on (X,d_1). Let $g\colon [-1,1]\to\mathbb{R}$ be the zero function, and define $f_n\colon [-1,1]\to\mathbb{R}$ to be the piecewise-linear function with the following graph.



Then $d_1(f_n,g)=\int_{-1}^{1} f_n(t)\,dt=\frac{1}{n}$, so $f_n\to g$ in (X,d_1) as $n\to\infty$. But $\mathrm{ev}_0(f_n)=1$, and $\mathrm{ev}_0(g)=0$, so $\mathrm{ev}_0(f_n)\not\to\mathrm{ev}_0(g)$ as $n\to\infty$. Thus ev_0 is not continuous at g.

On the other hand, ev_0 *is* a continuous function on (X,d_∞). Let $f\in X$ be arbitrary. We need to show that ev_0 is continuous at f. Let $\epsilon>0$ be arbitrary. Set $\delta=\epsilon$. Then for all $g\in X$ such that $d_\infty(f,g)<\delta$, we have $|\mathrm{ev}_0(f)-\mathrm{ev}_0(g)|=|f(0)-g(0)|\leq\sup_{t\in[-1,1]}|f(t)-g(t)|<\epsilon$. So ev_0 is continuous at f.