M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

PROBLEM SHEET 1

Exercise 1. Let p be a prime number. Define a function $d\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ \frac{1}{r} & \text{if } m \neq n, \text{ where } m - n = p^{r-1}q \text{ with } q \in \mathbb{Z} \text{ not divisible by } p. \end{cases}$$

Show that d is a metric on $\mathbb{Z}$.

Solution. We need to prove:

M1: $d(m,n) \geq 0$ with equality iff $m = n$;
M2: $d(m,n) = d(n,m)$ for all n, $m \in \mathbb{Z}$;
M3: $d(l,n) \leq d(l,m) + d(m,n)$ for all l, m, $n \in \mathbb{Z}$.

M1 is obvious. M2 is straightforward. For M3, let $l - m = p^{r-1}q$ and $m - n = p^{s-1}q'$, with q and q' coprime to p. Then $d(l,m) = 1/r$, $d(m,n) = 1/s$, and $l - n = p^{t-1}q''$ with q'' coprime to p and $t \geq \min(r,s)$. So $d(l,n) \leq 1/\min(r,s) \leq 1/r + 1/s$, which is (M3).

Exercise 2. Let $C([a,b])$ denote the set of continuous functions from $[a,b]$ to $\mathbb{R}$, and let $C^1([a,b])$ denote the set of differentiable functions $f\colon [a,b] \to \mathbb{R}$ such that f' is continuous. Let:

$$d_\infty(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

This defines a metric on both $C([a,b])$ and $C^1([a,b])$.

(1) Consider the map:

$$\begin{aligned} \textsc{Int} : \big(C([a,b]), d_\infty\big) &\to \big(C^1([a,b]), d_\infty\big) \\ f &\mapsto \int_a^x f(t)\,dt \end{aligned}$$

Is $\textsc{Int}$ continuous?

(2) Consider the map:

$$\begin{aligned} \textsc{Diff} : \big(C^1([a,b]), d_\infty\big) &\to \big(C([a,b]), d_\infty\big) \\ f &\mapsto f' \end{aligned}$$

Is $\textsc{Diff}$ continuous?

Solution. $\textsc{Int}$ is continuous. We have that

$$\begin{aligned} d_\infty\Big(\textsc{Int}(f), \textsc{Int}(g)\Big) &= \sup_{x\in[a,b]} \left| \int_a^x f(t) - g(t)\,dt \right| \\ &\leq \sup_{x\in[a,b]} \left(|x-a| \sup_{y\in[a,x]} |f(t) - g(t)| \right) \\ &\leq |b-a|\, d_\infty(f,g) \end{aligned}$$

So for any $\epsilon > 0$, if we set $\delta = \epsilon/(b-a)$ then $d_\infty\big(\textsc{Int}(f), \textsc{Int}(g)\big) < \epsilon$ whenever $d_\infty(f,g) < \delta$.

DIFF is not continuous. For example, let us assume for simplicity that $0 \in [a,b]$. Set g to be the zero function and, for any $\delta > 0$, set $f(x) = \delta \sin(x/\delta)$. Then $f \in B_\delta(g)$, but

$$d_\infty\Big(\text{DIFF}(f), \text{DIFF}(g)\Big) = \sup_{x\in[a,b]} |\cos(x/\delta)| = 1$$

So DIFF is not continuous at g.

Exercise 3. Let (X,d) be a metric space and A be a subset of X. Show that $x \in \partial A$ if and only if, for all $\epsilon > 0$, we have that $B_\epsilon(x) \cap A$ and $B_\epsilon(x) \cap (X \setminus A)$ are both non-empty.

Solution. Recall that $\partial A = \overline{A} \setminus \mathring{A}$. Suppose that $x \in \partial A$. Then $x \in \overline{A}$, and hence $B_\epsilon(x) \cap A$ is non-empty for all $\epsilon > 0$. Also $x \notin \mathring{A}$, so there does not exist $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. Put differently: $B_\epsilon(x) \cap (X \setminus A)$ is non-empty for all $\epsilon > 0$. We have shown that if $x \in \partial A$ then, for all $\epsilon > 0$, both $B_\epsilon(x) \cap A$ and $B_\epsilon(x) \cap (X \setminus A)$ are non-empty.

For the converse, the fact that $B_\epsilon(x) \cap A$ is non-empty for all $\epsilon > 0$ implies that $x \in \overline{A}$. And the fact that $B_\epsilon(x) \cap (X \setminus A)$ is non-empty for all $\epsilon > 0$ shows that there is no $\epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. In other words, $x \notin \mathring{A}$. So $x \in \overline{A} \setminus \mathring{A} = \partial A$.

Exercise 4. Let (X,d) be a metric space and A be a subset of X. For $x \in X$, define

$$d(x,A) = \inf\{d(x,a) : a \in A\}$$

Show that:

(1) $d(x,A) = 0$ if and only if $x \in \overline{A}$.
(2) for all $y \in X$, $d(x,A) \le d(x,y) + d(y,A)$.
(3) the map $x \mapsto d(x,A)$ defines a continuous function from X to $\mathbb{R}$.

Solution. (1) Suppose that $d(x,A) = 0$. Let (a_n) be a sequence of points in A such that $d(x,a_n) < 1/n$. Then, for all $\epsilon > 0$, the ball $B_\epsilon(x)$ contains some $a_n \in A$, and so $x \in \overline{A}$. Conversely, if $x \in \overline{A}$ then we can find a sequence (a_n) of points of A such that $a_n \in B_{1/n}(x)$, and so $\inf\{d(x,a) : a \in A\} \le \inf\{d(x,a_n) : n \in \mathbb{N}\} = 0$. Thus $d(x,A) = 0$.

(2) For all $a \in A$ we have

$$d(x,a) \le d(x,y) + d(y,a)$$

Thus

$$\inf_{a'\in A} d(x,a') \le d(x,y) + d(y,a)$$

or in other words

$$d(x,A) \le d(x,y) + d(y,a).$$

This holds for all $a \in A$, so therefore

$$d(x,A) \le \inf_{a\in A}\Big(d(x,y) + d(y,a)\Big) = d(x,y) + d(y,A).$$

(3) From part (2) we have that

$$d(x,A) - d(y,A) \le d(x,y)$$

and, switching x and y, also that

$$d(y,A) - d(x,A) \le d(x,y)$$

Thus $|d(x,A) - d(y,A)| \le d(x,y)$. We will show that the function $f\colon x \mapsto d(x,A)$ is continuous at y, where $y \in X$ is arbitrary. Let $\epsilon > 0$ be arbitrary, and let $\delta = \epsilon$. Whenever $d(x,y) < \delta$ we have that $|d(x,A) - d(y,A)| \le d(x,y) < \epsilon$. Thus f is continuous at y. But y was arbitrary, so f is continuous.