M2PM5 METRIC SPACES AND TOPOLOGY SPRING 2017

PROBLEM SHEET 2

Exercise 1

(1) Say $\epsilon := d(x, y)$. Define $U := \mathcal{B}(x, \epsilon/3)$, $V := \mathcal{B}(y, \epsilon/3)$, where for $x \in X$ and $\delta > 0$, $\mathcal{B}(x, \delta)$ denotes the open ball of radius δ around x. The sets U and V are open and contain x and y respectively. Also it is obvious that $U \cap V = \emptyset$.

Remark: some people asked what happens when you consider the discrete metric defined by $d(x, y) = 1$ if $x \neq y$, $d(x, y) = 0$ if $x = y$ on X. We claim that in this case the above is trivially true as we can take $U := \{x\}$ and $V := \{y\}$. Indeed under the discrete metric, singletons are open. (Notice that for any $0 < \delta < 1$, $\mathcal{B}(x, \delta) = \{x\}$ in this metric!).

(2) By definition, B bounded means that there exists $R > 0$ s.t. $B \subset \mathcal{B}(0, R)$. Thus $C \subset \mathcal{B}(0, R)$, which means that C is also bounded. For the diameters, we have

$$\begin{aligned} \operatorname{diam} C &:= \sup\{|x - y|; x, y \in C\} && (1)\\ &\leq \sup\{|x - y|; x, y \in B\} \text{ (taking the sup on a smaller set)} && (2)\\ &=: \operatorname{diam} B. && (3) \end{aligned}$$

Exercise 2

The function f is clearly continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$ as a product of continuous functions on $\mathbb{R}^2 \setminus \{(0, 0)\}$. We claim f is not continuous at $(0, 0)$. To prove this, one can exhibit two sequences that converge to a different limit. Alternatively (but in the same spirit), one can write $x = r\cos(\theta), y = r\sin(\theta)$ in polar co-ordinates and remark that $f(x, y) = \cos(\theta)\sin(\theta)$ is independent of r. Taking sequences along different angles give different results, hence f cannot be continuous at $(0, 0)$.

Exercise 3

(1) By definition of the closure, $\subseteq$ is clear (the RHS is a finite union of closed sets, hence it is closed and it clearly contains the LHS). We prove $\supseteq$. Say $x \in \bigcup_{i=1}^{m} \overline{A_i}$.

Then there exists $i \in \{1, \dots, m\}$ such that $x \in \overline{A_i}$, i.e., there exists a sequence $\{x_n\} \subseteq A_i$ converging to x. But this shows that there exists a sequence $\{x_n\} \subseteq \bigcup_{i=1}^m A_i$ converging to any $x \in \bigcup_{i=1}^m \overline{A_i}$, hence $RHS \subseteq LHS$.

(2) RHS is closed as an intersection of closed sets. Furthermore, it clearly contains $\bigcap_{i=1}^m A_i$. So it contains the smallest closed set with this property, which is by definition $\overline{\bigcap_{i=1}^m A_i}$.

(3) Take e.g. $A := (0,1)$, $B = (1,2)$ in $\mathbb{R}$. Then $\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$ while $\overline{A \cap B} = \overline{\emptyset} = \emptyset$.

Exercise 4 The closures are computed in $\mathbb{R}$!

(1) $\overline{[1,\infty)} = [1,\infty)$. ($[1,\infty)$ is closed in $\mathbb{R}$).
(2) $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$. (Every real is the limit of a sequence of irrationals.)
(3) $A := \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} =: \{x_n\}_{n\in\mathbb{N}}$ has one limit point in $\mathbb{R}$, namely $x = 1$ so $\overline{A} = A \cup \{x\}$.
(4) $A := \left\{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\right\} \cup \{0,1,2\}$. A is closed (it contains its only limit point 0). So $\overline{A} = A$.

Exercise 5

(1) At every step, each closed interval constituting C_n is split into two closed intervals. So the result follows by induction.

(2) C is closed as an intersection of closed sets.

(3) We have $K_n := [0, 3^{-n}] \in C_n$ for all $n \geq 0$. The K_n form a nested sequence of compact sets, hence their intersection is non-empty (alternatively, it is obvious that $\bigcap_{n\geq 0} K_n = \{0\}$ so $0 \in C$). Clearly, $\bigcap_{n\geq 0} K_n \subseteq \bigcap_{n\geq 0} C_n =: C$.

(4) We exhibit a surjection $C \twoheadrightarrow [0,1]$. This shows that C has at least the cardinality of the reals, hence is uncountable. For this, remark that $x \in C_n$ iff $x = \overline{0.a_1 \dots a_n \dots}^3$, where $a_i \in \{0,2\}$ for all $i \in \{1,\dots,n\}$. So $x \in C$ iff $x = \overline{0.a_1a_2a_3\dots}^3$ with all $a_i \in \{0,2\}$. Consider the map $\Phi : C \to [0,1]$ defined by

$$\Phi(x) := \sum_{i=1}^{\infty} \frac{a_i}{2} 2^{-i} =: \overline{0.b_1b_2\dots}^2,$$

where $x = \sum_{i=1}^{\infty} a_i 3^{-i} \in C$ and $b_i := a_i/2 \in \{0,1\}$. Clearly, Φ is onto (as we get all the possible binary expansions in [0,1]). So the result follows.

(5) We prove more than needed: we show that each point of C is both an accumulation point of C^c (the complement of C) and an accumulation point of C.

By (4), we know that $x \in C$ iff there are no 1's in its triadic expansion. Say $x \in C$, $x = \overline{0.a_1a_2a_3\ldots}^3$. Suppose there are infinitely non-zero a_i. Then define $x_n := \overline{0.a_1a_2a_3\ldots a_n0\ldots}^3$. Clearly $x_n \in C$, $x_n \neq x$ for all n and $x_n \to x$. If there are only finitely many non zero a_i, write a_k for the largest non-zero digit. Then define $x_n := \overline{0.a_1\ldots a_k0\ldots2\ldots}^3$, where the 2 is the $k+n$-th position. Again $x_n \in C$, $x_n \neq x$ for all n and $x_n \to x$. So this proves that any point of C is an accumulation point in C.

Recall $C = \bigcap_{n\geq 0} C_n$ and say $x \in C$. Let $\epsilon > 0$. We want to show that there exists $y \notin C$ such that $|x-y| < \epsilon$. Take $n > -\frac{\log(\epsilon)}{\log(3)}$. Then, x is contained in an interval of length 3^{-n}. Split the interval in three parts. We know that the middle part is in C^c by construction. So take y in this part. We have $y \in C^c$ and $|x-y| < 3^{-n} < \epsilon$, which shows that any point of C is an accumulation point for C^c.

(6) Say $x \in C$ and let $\epsilon > 0$. We claim that $\mathcal{B}(x,\epsilon) \cap C^c \neq \emptyset$ (so C is nowhere dense). This follows from (5): we have seen that any point of C is an accumulation point for C^c so there exists $y \in C^c$ such that $|x-y| < \epsilon$, i.e., $y \in \mathcal{B}(x,\epsilon)$, which is what we wanted.