M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

SOLUTIONS TO PROBLEM SHEET 3

Exercise 1.

(1) Let $A \subset X$ be a finite set. We claim that A is closed and therefore $\overline{A} = A$. To check this it suffices to see that $X \setminus A$ is open since $X \setminus (X \setminus A) = A$ which is finite.

(2) Let $A \subset X$ be an infinite set. We claim that $\overline{A} = X$ in this case. To show this we want to show the only closed set containing A is X and therefore the intersection of all closed sets containing A is X. Suppose $A \subseteq B$ where B is a closed set. Then $X \setminus B$ is open. Since $X \setminus (X \setminus B) = B$ and B contains the infinite A, we must have $X \setminus B = \varnothing$ hence $X = B$ as required.

Exercise 2. We prove that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X but give a counterexample to show $\mathcal{T}_1 \cup \mathcal{T}_2$ need not be a topology on X.

To show $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X we need to check the three topological space axioms.

(T1) We have $\varnothing, X \in \mathcal{T}_1, \mathcal{T}_2$ by definition. Therefore $\varnothing, X \in \mathcal{T}_1 \cap \mathcal{T}_2$.

(T2) Take an arbitrary collection of open sets $\{U_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$. Then $\bigcup\limits_{\lambda} U_\lambda \in \mathcal{T}_1, \mathcal{T}_2$ and therefore $\bigcup\limits_{\lambda} U_\lambda \in \mathcal{T}_1 \cap \mathcal{T}_2$.

(T3) Take a finite collection of open sets $U_1, \ldots, U_n \in \mathcal{T}_1, \mathcal{T}_2$. Then $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}_1, \mathcal{T}_2$ and therefore $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}_1 \cap \mathcal{T}_2$.

To show $\mathcal{T}_1 \cup \mathcal{T}_2$ need not be a topology on X, let $X = \{1, 2, 3, 4\}$. Let $\mathcal{T}_1 = \{\varnothing, X, \{1\}, \{2\}, \{1, 2\}\}$ and $\mathcal{T}_2 = \{\varnothing, X, \{1\}, \{3\}, \{1, 3\}\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}$. Now we can see that $\{1, 2\}$ and $\{1, 3\}$ are in $\mathcal{T}_1 \cup \mathcal{T}_2$ but $\{1, 2, 3\} = \{1, 2\} \cup \{1, 3\}$ is not.

Exercise 3. By definition $\partial A = \overline{A} \setminus \mathring{A}$. First note that $B \setminus C = B \cap (X \setminus C)$. We also have that $\mathring{A} = X \setminus \overline{X \setminus A}$ so $X \setminus \mathring{A} = \overline{X \setminus A}$. Therefore $\partial A = \overline{A} \cap (X \setminus \mathring{A}) = \overline{A} \cap \overline{X \setminus A}$ as required.

Exercise 4.

(1) We show p_1 is continuous, an analogous argument shows that p_2 is continuous. Given U open in X we must show $p_1^{-1}(U) \in \mathcal{T}$. By definition of p_1 we see that $p_1^{-1}(U) = U \times Y$ and since U is open in X and Y is open in Y we see that $U \times Y \in \mathcal{T}$.

(2) Assume $f : Z \to X \times Y$ is continuous. Then $p_1 \circ f$ and $p_2 \circ f$ are continuous because they are compositions of continuous functions.

Now we assume $p_1 \circ f$ and $p_2 \circ f$ are continuous. Let W be open in $X \times Y$. Then W is a union of elements of $\mathcal{B}$, say $W = \bigcup\limits_{\lambda \in \Lambda} U_\lambda \times V_\lambda$. Since arbitrary unions of open sets are open in Z it suffices to check that $f^{-1}(U_\lambda \times V_\lambda)$ is open in Z for every $\lambda \in \Lambda$. Let $(p_1 \circ f)^{-1}(U_\lambda) = Z_1$ and $(p_2 \circ f)^{-1}(V_\lambda) = Z_2$. Then Z_1 and Z_2 are open in Z because $p_1 \circ f$ and $p_2 \circ f$ are continuous. By definition of Z_1 and Z_2 we must have $f^{-1}(U_\lambda \times V_\lambda) \supseteq Z_1 \cap Z_2$ and since any element in Z which maps into $U_\lambda \times V_\lambda$ must map into U_λ under $p_1 \circ f$ and map into V_λ under $p_2 \circ f$ we have $f^{-1}(U_\lambda \times V_\lambda) \subseteq Z_1 \cap Z_2$. Hence $f^{-1}(U_\lambda \times V_\lambda) = Z_1 \cap Z_2$ which is open in Z, as required.