# M2PM5 METRIC SPACES AND TOPOLOGY
# SPRING 2017

SOLUTIONS TO PROBLEM SHEET 4

**Exercise 1.**

(1) If $X$ is compact then every open cover has a finite subcover. Since $X$ has the discrete topology the set of all singletons form an open cover of $X$, $U_x = x$ for all $x \in X$. The only subcover of this is the cover itself. This is because removing any set, $U_x$ will mean $x$ is no longer in the the union of the sets. Hence as $X$ is compact we must have that this cover is in fact finite, and hence $X$ must be finite.

Conversely, if $X$ is finite then any open cover of $X$ (after deleting any repeats of the same set) will contain at most $2^{|X|}$ sets and is therefore finite.

(2) Take an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of $Y \cup Z$. Then $\{U_\alpha\}$ is an open cover of $Y$ so there exists a finite subset $A \subseteq \Lambda$ such that $\{U_\alpha\}_{\alpha\in A}$ is an open cover for $Y$. Similarly there exists a finite subset $B \subseteq \Lambda$ such that $\{U_\alpha\}_{\alpha\in B}$ is an open cover for $Z$. Therefore $A \cup B$ is finite and $\{U_\alpha\}_{\alpha\in A\cup B}$ is an open finite subcover for $Y \cup Z$ as required.

**Exercise 2.** Take an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of $f(X) \subseteq Y$. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ is an open cover of $X$. Since $X$ is compact there is a finite subset $A \subseteq \Lambda$ such that $\{f^{-1}(U_\alpha)\}_{\alpha\in A}$ is an finite cover of $X$. Then $\{U_\alpha\}_{\alpha\in A}$ is an open cover of $f(X)$ (it certainly consists of open sets, and if $y \in f(X)$ then $y = f(x)$ for some $x \in X$, and since $x \in f^{-1}(U_\alpha)$ for some $\alpha \in A$ we have that $y \in U_\alpha$.). Therefore every open cover of $f(X)$ has a finite subcover as required.

**Exercise 3.** Suppose the intersection is empty and let $U_n = X \setminus V_n$. Then $U_n$ is open for each $n$ and $U_j \subseteq U_{j+1}$. Moreover $\bigcup U_n = \bigcup(X \setminus V_n) = X \setminus \bigcap V_n = X$. Therefore by the compactness of $X$ we only need finitely many of these, so $X = U_1 \cup \cdots \cup U_m$ for some $m$. Since $U_j \subseteq U_{j+1}$ it follows that $X = U_m$. However $U_m = X \setminus V_m$ which would imply $V_m$ is empty, a contradiction.

The statement is not true without the assumption of compactness. Let $X = (0, 1]$ with the usual topology induced from $\mathbb{R}$. Then $X$ is not compact. Let $V_n = (0, \frac{1}{n+1}]$, which are closed in $X$. Then clearly $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ but $\bigcap\limits_{n\geq 0} V_n = \varnothing$ (we can of course take a large enough $N$ such that any real number strictly greater than 0 is not contained in $V_n = (0, \frac{1}{N+1}]$).

**Exercise 4.** Suppose that $A \cup B$ is disconnected. Then there exists a continuous surjective map $f : A \cup B \to \{0, 1\}$. Consider the restriction of $f$ to $A$, $f|_A : A \to \{0, 1\}$. Then $f|_A$ is continuous but cannot be surjective as $A$ is connected. Therefore we may assume $f(a) = 0$ for all $a \in A$. Similarly $f|_B$ is a continuous map from $B$ to $\{0, 1\}$ which cannot be surjective. If $f(b) = 0$ for all $b \in B$ then $f$ would not be surjective so we must have that $f(b) = 1$ for all $b \in B$. So it is clear that $f^{-1}(0) = A$ which must be open in $A \cup B$ as $f$ is continuous. We know that $A \cap \overline{B} \neq \varnothing$ so there exists $a \in A$ such that $a \in \overline{B}$, therefore every neighbourhood of $a$ contains an element of $B$. Now $f^{-1}(0)$ is a neighbourhood of $a$ and therefore must contain some $b \in B$. This is a contradiction though, because $f(b) = 1$ for all $b \in B$. Therefore $A \cup B$ is connected as required.

Of course we cannot drop the assumption $A \cap \overline{B} \neq \varnothing$. Letting $X = \mathbb{R}$ with the usual topology and $A = (0, 1)$, $B = (1, 2)$ gives an easy counterexample.