M2PM5 METRIC SPACES AND TOPOLOGY SPRING 2017

PROBLEM SHEET 5

January 3, 2017

Exercise 1

(1) We check the axioms:

(T1) $\emptyset \in \mathcal{T} \subset \tilde{\mathcal{T}}$.

(T2) As regards finite intersections, the intersection of $V \cup \{\infty\}$ and $W \cap \{\infty\}$ is again a set of the right form since a finite union of compact sets is compact. The intersection of a set $U \in \mathcal{T}$ and a set $V \cup \{\infty\}$ is $U \cap V$, where $V \in \mathcal{T}$ so $U \cap V \in \mathcal{T} \subset \tilde{\mathcal{T}}$. Finally, the intersection of two sets in $\mathcal{T}$ is in $\mathcal{T}$ hence in $\tilde{\mathcal{T}}$.

(T3) Suppose that $U_i \in \tilde{\mathcal{T}}$, $i \in I$. We need to show that $\bigcup_{i \in I} U_i \in \tilde{\mathcal{T}}$. Either each U_i is in $\mathcal{T}$ – in which case $\bigcup_{i \in I} U_i \in \mathcal{T} \subset \tilde{\mathcal{T}}$ because $\mathcal{T}$ is a topology – or we can write $I = J \cup K$ with:

* $U_j \in \mathcal{T}$ for all $j \in J$;
* $U_k = V_k \cup \{\infty\}$ with $X \setminus V_k$ compact and closed, for all $k \in K$; and
* K non-empty.

Then $\bigcup_{i \in I} U_i = X \setminus V$ where $V = X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} X \setminus U_i$. Now

$$V = \bigcap_{i \in I} X \setminus U_i = \left(\bigcap_{j \in J} X \setminus U_j \right) \cap \left(\bigcap_{k \in K} V_k \right)$$

The left-hand set here is an intersection of closed sets, hence is closed. The right-hand set here is an intersection of closed sets, hence is closed, and is a subset of a compact set V_k (recall that there exists some $k \in K$). Thus the right-hand set here is a closed subset of a compact set, hence compact, and so V is a closed subset of a compact set. Hence V is compact; thus $\bigcup_{i \in I} U_i \in \tilde{\mathcal{T}}$.

(2) Let $\{U_\alpha\}$ be an open cover of $\tilde{X}$. Take any set in the cover that contains $\{\infty\}$, say $U^* := V \cup \{\infty\}$. Then $X \setminus V$ is compact and closed. $\{U_\alpha \cap (X \setminus V)\}$ is an open cover of $X \setminus V$. Hence we can take a finite sub-cover, say $U_{\alpha_1} \cap (X \setminus V), \ldots, U_{\alpha_n} \cap (X \setminus V)$. Whence, $U^*, U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite sub-cover of $\tilde{X}$.

(3) The subspace topology $\tilde{\mathcal{T}} \cap X$ consists of $\mathcal{T}$ and the sets $V \subseteq X$ such that $X \setminus V$ is compact and closed. These V's are thus in $\mathcal{T}$ and the subspace topology on X coincides with $\mathcal{T}$.

(4) $(\tilde{X}, \tilde{\mathcal{T}})$ is homeomorphic to $\mathbb{S}^1$, the unit sphere in $\mathbb{R}^2$. A homeomorphism is given by the stereographic projection $\phi : \mathbb{S}^1 \to \tilde{X}$ defined by

$$\phi(x, y) := \frac{x}{1-y}, \ \phi(0, 1) = \infty. \tag{1}$$

It should be clear that the map is bijective and continuous with respect to the natural subspace topology on $\mathbb{S}^1$ and the topology $\tilde{\mathcal{T}}$ on $\tilde{X}$. The inverse map is given by

$$\phi^{-1}(t) = (\frac{2t}{1+t^2}, \frac{-1+t^2}{1+t^2}) \tag{2}$$

and is also continuous with respect to these topologies. Notice that the topology $\tilde{\mathcal{T}}$ is exactly the topology needed to make ϕ and ϕ^{-1} continuous. Therefore the stereographic projection $\phi : \mathbb{S}^1 \to \tilde{X}$ is a homeomorphism, i.e. $\mathbb{S}^1 \approx \tilde{\mathcal{T}}$.

(5) $(\tilde{X}, \tilde{\mathcal{T}})$ is homeomorphic to the sphere $\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$. The homeomorphism is again given by the stereographic projection $\phi : \mathbb{S}^2 \to \tilde{X}$ defined by

$$\phi(x, y, z) := (\frac{x}{1-z}, \frac{y}{1-z}). \tag{3}$$

The inverse map is given by

$$\phi^{-1}(t, u) := (\frac{2t}{1+t^2+u^2}, \frac{2u}{1+t^2+u^2}, \frac{-1+t^2+u^2}{1+t^2+u^2}). \tag{4}$$

Once again it is easy to check that ϕ and ϕ^{-1} are continuous hence, i.e. $\mathbb{S}^2 \approx \tilde{\mathcal{T}}$.

Exercise 2

(1) Obvious from a picture. Perhaps easier to formalize in polar co-ordinates. We exhibit a path that works. Say $P_1 = (r_1, \theta_1)$, $P_2 = (r_2, \theta_2)$. Consider the path γ defined by $\gamma(t) = (r_2, \theta_2 + 2t(\theta_1 - \theta_2))$ for $t \in [0, 1/2]$ and $\gamma(t) = (r_2 + 2(t - 1/2)(r_2 - r_1), \theta_1)$ for $t \in [1/2, 1]$. γ is obviously continuous and lies in X. Furthermore, $\gamma(0) = P_2$ and $\gamma(1) = P_1$ so we are done.

(2) Again this is clear by drawing a picture. There is no path from $(-1, 0)$ to $(2, 0)$ laying in X. For if there be such a path, $f = p_1 \circ \gamma$ (p_1 being the projection on the x-axis) would be a continuous functions with value in $\mathbb{R}$ satisfying $f(0) = -1$ and $f(1) = 2$. Hence, by the intermediate value theorem, there would exist $t \in [0, 1]$ such that $f(t) = 0.5$, which contradicts the fact that γ lies in X.

Exercise 3

(1) The Wikipedia page for the Topologist's Sine Curve has a good picture.

(2) Use Exercise 4 on problem sheet 4.

(3) If T was path connected, there would exist a path $\gamma : [0, 1] \to T$ such that $\gamma(0) = (1, \sin(1))$ and $\gamma(1) = (0, 0)$, say. WLOG $\gamma(t) = (x(t), \sin(1/x(t)))$ for $t \in [0, 1)$, with $x(\cdot)$ continuous, $x(t) \to 0$ as $t \to 1$. Say $\epsilon = 10^{-99}$, γ being continuous, we would need to have $||\gamma(t)|| = ||\gamma(t) - \gamma(1)|| < \epsilon$ for all t close enough to 1 i.e. $|\sin(x^{-1})| \leq ||(x, \sin(x^{-1})|| < \epsilon$ for all x close enough to 0. But that is clearly not the case.