Solutions to Problem Sheet 8

Exercise 1. In all three cases we compute the pointwise limit of the sequence $\{f_n\}$, say f. If this is not continuous then we know that the convergence cannot be uniform (uniform convergence of continuous functions implies the limit is continuous). If it is then we must compute $M_n = \sup_{x\in[0,1]}|f_n(x) - f(x)|$ and check whether $M_n \to 0$ as $n \to \infty$. If it does then the convergence is uniform, if not then the convergence is not uniform.

(1) The pointwise limit on $[0, 1]$ is the 0 function. If we rewrite $f_n(x)$ as $(\frac{1}{x} + n)^{-1}$ then it is easy to see that $M_n = \frac{1}{n+1}$ since the supremum is obtained at $x = 1$. Therefore $M_n \to 0$ as $n \to \infty$ then the convergence is uniform on $[0, 1]$.

(2) In this case the pointwise limit is the function f where $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = \frac{1}{2}$. Since f is not continuous the convergence of f_n cannot be uniform.

(3) First we need to find the pointwise limit. Clearly $f_n(0) = 0$ for all n. Now fix $x \in (0, 1]$ and notice that $n(1 - x^2)^{n^2} \to 0$ as $n \to \infty$ (because x is fixed and less than or equal 1). Therefore the pointwise limit is the zero function on $[0, 1]$. We now need to use some calculus to find M_n. We find the value of $x \in [0, 1]$ for which f_n attains its maximum value (this exists because f_n is continuous (in fact differentiable) and [0,1] is compact). To do this we can just differentiate f_n and find when the derivative is 0 (on $[0, 1]$). In this case $f_n'(x) = n(1 - x^2)^{n^2-1}(1 - (2n^2 + 1)x^2)$. From this we see that $x = 1$ or $\frac{1}{(2n^2+1)^{1/2}}$. Since $f_n(1) = 0$ we must have that

$$M_n = f_n\left(\frac{1}{(2n^2+1)^{1/2}}\right) = \frac{n}{(2n^2+1)^{1/2}}\left(1 - \frac{1}{(2n^2+1)}\right)^{n^2}.$$

We know need to compute the limit of M_n as n tends to infinity. To do this we just notice that we can rewrite the right hand side as follows.

$$\frac{n}{(2n^2+1)^{1/2}}\left(1 - \frac{1}{(2n^2+1)}\right)^{n^2} = \left(\frac{1}{(2+1/n^2)^{1/2}}\right)\left(1 - \frac{1}{(2n^2+1)}\right)^{n^2}.$$

Now we can see that the limit of M_n is the product of the limits of the two brackets, the left hand one being $\frac{1}{\sqrt{2}}$ and the right hand one being $e^{-\frac{1}{2}}$. Therefore the limit is $\frac{1}{\sqrt{2e}} \neq 0$ so the convergence is not uniform.

Exercise 2. Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \frac{1}{n} & \text{if } x > 0. \end{cases}$$

Then the pointwise limit of the f_ns is the zero function on $\mathbb{R}$, which is continuous, and the convergence is uniform since $1/n \to 0$ as $n \to \infty$.

Exercise 3. Let $X = [1, +\infty)$ and $f\colon X \to X$ defined by $f(x) = x + x^{-1}$.

(1) X is a closed subspace of $\mathbb{R}$, which is complete so by lectures, X is complete. (See solution of Exercise 17.2 in the second coursework.)

(2) Let $x, y \in X$, $x \neq y$. We have

$$\begin{aligned} |f(x) - f(y)| &= |x + x^{-1} - y - y^{-1}| \\ &= \left| x - y + \frac{y - x}{xy} \right| \\ &= |x - y| \cdot \left| 1 - \frac{1}{xy} \right| \\ &< |x - y| \end{aligned}$$

as $1 - \frac{1}{xy} \in (0, 1)$ for $x, y \in X$. Remark that this does NOT mean that f is a contraction.

(3) Let $x \in X$. Since $f(x) - x = x^{-1} \neq 0$, we have $f(x) \neq x$.

Exercise 4. Let X be a topological space, let $x_0 \in X$ be a point. We consider the set of loops based at the point x_0, i.e. continuous maps $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$. Two loops γ_1 and γ_2 are called equivalent, and we write $\gamma_1 \sim \gamma_2$, if γ_1 and γ_2 are homotopic relatively to $\{0, 1\}$. (See solution to Exercise 5.2 in Problem Sheet 7.) Via the same method of the solution of Exercise 4 in Problem Sheet 6, one can show that $\sim$ is an equivalence relation on the set of loops based at x_0. We denote by $\pi_1(X, x_0)$ the quotient set, i.e. the set of equivalence classes, and by $[\gamma] \in \pi_1(X, x_0)$ the equivalence class of the loop γ.

If γ_1 and γ_2 are two loops based at x_0, we define the $\gamma_1 * \gamma_2$ by $\gamma_1 * \gamma_2\colon [0, 1] \to X$

$$(\gamma_1 * \gamma_2)(t) \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is easy to check that $\gamma_1 * \gamma_2$ is a loop based at x_0. We need to prove that this operation induce a well defined operation on $\pi_1(X, x_0)$ by setting

$$[\gamma_1] * [\gamma_2] := [\gamma_1 * \gamma_2]$$

for all γ_1 and γ_2. In order to do that, we have to check that the equivalence class of $\gamma_1 * \gamma_2$ does not change if we replace γ_i with an equivalent loop, for both $i = 1$ and $i = 2$. So, suppose now that we have four loops $\gamma_1, \delta_1, \gamma_2, \delta_2$ based at x_0 such that $\gamma_1 \sim \delta_1$ and $\gamma_2 \sim \delta_2$. For $i = 1, 2$, let H_i be a homotopy between γ_i and δ_i, i.e. $H_i\colon [0, 1] \times [0, 1] \to X$ is a continuous map such that $H_i(\cdot, 0) = \gamma_i$, $H_i(\cdot, 1) = \delta_i$, $H_i(0, \cdot) = H_i(1, \cdot) = x_0$; then consider $H\colon [0, 1] \times [0, 1] \to X$ defined by

$$H(t, s) = \begin{cases} H_1(2t, s) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ H_2(2t - 1, s) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

One can see that H is well defined because $H_1(1, s) = x_0 = H_2(0, s)$ for any $s \in [0, 1]$. Since we are gluing continuous functions on the finite closed cover $\{[0, 1/2] \times [0, 1], [1/2] \times [0, 1]\}$ of the square $[0, 1] \times [0, 1]$, we have that H is continuous by the pasting lemma. Moreover, it is clear that $H(0, \cdot) = H(1, \cdot) = x_0$, $H(\cdot, 0) = \gamma_1 * \gamma_2$ and $H(\cdot, 1) = \delta_1 * \delta_2$. In other words, H is a homotopy between $\gamma_1 * \gamma_2$ and $\delta_1 * \delta_2$. Therefore $\gamma_1 * \gamma_2 \sim \delta_1 * \delta_2$.

Identity element. Let $e\colon [0, 1] \to X$ be the constant loop based at x_0, i.e. $e(t) = x_0$ for any $t \in [0, 1]$. We have to prove that $[e]$ is the identity element of $\pi_1(X, x_0)$, i.e. $[\gamma] * [e] = [e] * [\gamma] = [\gamma]$ for any element $[\gamma] \in \pi_1(X, x_0)$. Equivalently, we need to show that $\gamma * e \sim e * \gamma \sim \gamma$ for any γ loop based at x_0. We see that

$$(e * \gamma)(t) = \begin{cases} x_0 = \gamma(0) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \gamma(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Consider the function $\psi\colon [0,1] \to [0,1]$ defined by

$$\psi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2t-1 & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice that ψ is continuous. We see that $e*\gamma = \gamma \circ \psi$. Consider the map $H\colon [0,1] \times [0,1] \to X$ defined by $H(s,t) = \gamma((1-s)t + s\psi(t))$ for any $t, s \in [0,1]$. Since H is built from compositions and sums of continuous functions, it is continuous. Moreover, $H(0,s) = \gamma(0) = x_0$ and $H(1,s) = \gamma(1) = x_0$ for any $s \in [0,1]$, $H(\cdot, 0) = \gamma$ and $H(\cdot, 1) = \gamma \circ \psi = e * \gamma$. This means that H is a homotopy between γ and $e * \gamma$. Therefore $e * \gamma \sim \gamma$.

Proving that $\gamma * e \sim \gamma$ is completely analogous. We consider the map $\eta\colon [0,1] \to [0,1]$ defined by

$$\psi(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We notice that $\gamma * e = \gamma \circ \psi$. We use the homotopy H defined by $H(s,t) = \gamma((1-s)t + s\eta(t))$.

Inverse. Let γ be a loop based at x_0. We denote by γ^{-1} the loop defined by $\gamma^{-1}(t) = \gamma(1-t)$ for any $t \in [0,1]$. We want to show that $\gamma * \gamma^{-1} \sim e$. Consider $H\colon [0,1] \times [0,1] \to X$ defined by

$$H(t,s) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq s/2 \\ \gamma(s) & \text{if } s/2 \leq t \leq 1 - s/2 \\ \gamma(2-2t) & \text{if } 1 - s/2 \leq t \leq 1. \end{cases}$$

One can check that H is well-defined and continuous (we have decomposed the square $[0,1] \times [0,1]$ into three triangles, this is a finite closed cover, so we can use the pasting lemma) and $H(t,0) = x_0 = e(t)$, $H(t,1) = (\gamma * \gamma^{-1})(t)$, $H(s,0) = x_0$, $H(s,1) = x_0$ for any $s, t \in [0,1]$. This implies that $\gamma * \gamma^{-1} \sim e$.

It is obvious to notice that $(\gamma^{-1})^{-1} = \gamma$. So, if we apply what we have proved to γ^{-1}, we get $\gamma^{-1} * \gamma \sim e$.

Associativity. Let α, β, γ be three loops based at x_0. We want to show that the two loops $\delta := \alpha * (\beta * \gamma)$ and $\varepsilon := (\alpha * \beta) * \gamma$ are equivalent. (Notice that these two loops are different!) We have

$$\begin{aligned} \delta(t) &= \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (\beta * \gamma)(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \\ &= \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(4t-2) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma(4t-3) & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases} \end{aligned}$$

and

$$\varepsilon(t) = \begin{cases} \alpha(4t) & \text{if } 0 \leq t \leq \frac{1}{4} \\ \beta(4t-1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Let us consider the function $\phi\colon [0,1] \to [0,1]$ defined by

$$\phi(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{4} \\ t + \frac{1}{4} & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is easy to see that ϕ is continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$. (Actually it is a homeomorphism of $[0,1]$ with itself.) One can check that $\varepsilon = \delta \circ \phi$. Now we consider the map $H\colon [0,1]\times[0,1]\to X$ defined by $H(t,s)=\delta(s\phi(t)+(1-s)t)$ for any $t,s\in[0,1]$. Since H is built from compositions and sums of continuous functions, it is continuous. Moreover $H(0,\cdot)=\delta(0)=x_0$, $H(1,\cdot)=\delta(1)=x_0$, $H(\cdot,0)=\delta$, $H(\cdot,1)=\delta\circ\phi=\varepsilon$. Therefore H is a homotopy between δ and ε.

Exercise 5. In order to show that given a continuous map $f\colon X\to Y$ with $f(x_0)=y_0$, we have that

$$f_*\colon \pi_1(X,x_0)\to\pi_1(Y,y_0)\qquad [\gamma]\to[f\circ\gamma]$$

is a well defined group homomorphism we have to show:

(1) if $\gamma_1\sim\gamma_2$ then $f\circ\gamma_1\sim f\circ\gamma_2$,
(2) $f_*([\gamma_1]*[\gamma_2])=f_*[\gamma_1]*f_*[\gamma_2]$,
(3) the identity $[e_{x_0}]$ gets mapped to the identity $[e_{y_0}]$.

None of this is difficult; in fact

(1) If $H(t,s)$ is a homotopy between $\gamma_1(t)$ and $\gamma_2(t)$, then $f\circ H$ is a homotopy between $f\circ\gamma_1$ and $f\circ\gamma_2$.
(2) This is also immediate, in fact the operations of composition of paths and inversion of paths are operations on the domain $I=[0,1]$ and so they commutes with composition with continuous functions. Indeed, just remember that $\gamma_1*\gamma_2\colon[0,1]\to X$ is defined by
$$(\gamma_1*\gamma_2)(t)=\begin{cases}\gamma_1(2t) & \text{if } 0\le t\le\frac{1}{2},\\ \gamma_2(2t-1) & \text{if } \frac{1}{2}\le t\le 1.\end{cases}$$
And it follows immediately that $f\circ(\gamma_1*\gamma_2)=(f\circ\gamma_1)*(f\circ\gamma_2)\colon[0,1]\to Y$.
(3) It's just by definition of f_*.

Remark. We have proved that given a topological space X and a base point $x_0\in X$ we can associate to (X,x_0) the group $\pi_1(X,x_0)$ of loops based in x_0 up to homotopy. Moreover we have seen that given a continuous map between two pointed topological spaces $f\colon(X,x_0)\to(Y,y_0=f(x_0))$ we have a homomorphism of groups $f_*\colon\pi_1(X,x_0)\to\pi_1(Y,y_0)$. It is moreover not at all difficult to see that: if we consider the identity $id_X\colon X\to X$, the induced morphism $(id_X)_*$ on $\pi_1(X,x_0)$ is the identity; given two continuous maps of pointed topological spaces

$$(X,x_0)\xrightarrow{f}(Y,y_0)\xrightarrow{g}(Z,z_0)$$

we have that $(g\circ f)_*=g_*\circ f_*\colon\pi_1(X,x_0)\to\pi_1(Z,z_0)$. This means that the fundamental group $\pi_1(-)$ is a so called functor from the category of pointed topological spaces to the category of groups.

If X is a path connected topological space, with arguments of the same flavour of those used in the last two exercises of this problem sheet, one can prove that $\pi_1(X,x)\cong\pi_1(X,x')$ for any $x,x'\in X$. In this case one usually talks about the fundamental group of X and writes $\pi_1(X)$.