M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2016

PROBLEM SHEET 6

Exercise 1. Let X denote the space of all continuous real-valued functions on $[a, b]$ equipped with the sup metric:

$$d(f, g) = \sup_{s \in [a,b]} |f(s) - g(s)|$$

Is X path-connected? Is X connected?

Exercise 2. (The Pasting Lemma) Let X and Y be topological spaces, let A and B be subsets of X such that $X = A \cup B$, and let $f\colon X \to Y$ be a map.

(1) Suppose that A and B are open, and that $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous. Show that f is continuous.

(2) Suppose that A and B are closed, and that $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous. Show that f is continuous.

Exercise 3. Let X be a topological space and let x_0, x_1 be points of X. Let γ_0, γ_1 be paths in X from x_0 to x_1. A *homotopy* from γ_0 to γ_1 is a continuous map $H\colon [0,1]\times[0,1]\to X$ such that:

- $H(s,0)=\gamma_0(s)$ for all $s\in[0,1]$;
- $H(s,1)=\gamma_1(s)$ for all $s\in[0,1]$;
- $H(0,t)=x_0$ for all $t\in[0,1]$;
- $H(1,t)=x_1$ for all $t\in[0,1]$.

You can think of a homotopy H as a family of paths γ_t, $t\in[0,1]$, defined by $\gamma_t(s)=H(s,t)$, which interpolates continuously between γ_0 and γ_1. The third and fourth conditions here say that each γ_t is a path from x_0 to x_1. We say that paths γ_0 and γ_1 are *homotopic* if and only if there exists a homotopy from γ_0 to γ_1.

(1) Show that any two paths in $\mathbb{R}^2$ with the same endpoints x_0 and x_1 are homotopic.
(2) Let D be a convex subset of $\mathbb{R}^n$. Show that any two paths in D with the same endpoints x_0 and x_1 are homotopic.

Exercise 4. Let γ_1 and γ_2 be paths in a topological space X. Write $\gamma_1\sim\gamma_2$ if and only if γ_1 is homotopic to γ_2. Show that $\sim$ is an equivalence relation.