M2PM5 METRIC SPACES AND TOPOLOGY
SPRING 2017

PROBLEM SHEET 7

Exercise 1. Prove that the open interval $(0, 1)$ is not sequentially compact.

Exercise 2. We say that a metric space X is *totally bounded* if and only if for each $\epsilon > 0$ there exist finitely many points $x_1, \ldots, x_N \in X$ such that:

$$B_\epsilon(x_1), \ldots, B_\epsilon(x_N)$$

is a cover of X. Show that a metric space X is compact if and only if it is complete and totally bounded.
(Hint: for one direction, show that complete and totally bounded implies sequentially compact.)

Exercise 3. Let X be a sequentially compact metric space and $V_1 \supseteq V_2 \supseteq \cdots$ be a sequence of nested closed subsets of X. Show that

$$\operatorname{diam}\left(\bigcap_{n=1}^{\infty} V_n\right) = \inf\left\{ \operatorname{diam}(V_n) : n = 1, 2, \dots \right\}.$$

Exercise 4. Recall the definition of *homotopy of paths* from Problem Sheet 6. Give an explicit homotopy between the following paths in $\mathbb{R}^2$:

$$\begin{aligned} f\colon [0,1] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \big(\cos \pi t, \sin \pi t\big) \end{aligned} \qquad\qquad \begin{aligned} g\colon [0,1] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \big(\cos \pi t, -\sin \pi t\big) \end{aligned}$$

Exercise 5. We say that a subspace D of $\mathbb{R}^n$ is *star-shaped* if and only if there exists a point $x_0 \in D$ such that, for all $x \in D$, the straight line segment from x_0 to x lies entirely within D. Let D be a star-shaped subspace of $\mathbb{R}^n$.

(1) Show that D is path-connected.
(2) Show that every path in D is homotopic to a constant path.