Solutions to Problem Sheet 6

Exercise 1. We need to show that for any $f, g \in C([a,b])$ there exists a continuous path

$$\gamma\colon [0,1] \to C([a,b])$$

such that $\gamma(0) = g$ and $\gamma(1) = f$. Let us define γ as follows

$$\gamma(t) = t \cdot f + (1-t) \cdot g.$$

As we wanted $\gamma(0) = g$ and $\gamma(1) = 1$ so we are just left to prove that the path is continuous at any $t_0 \in [0,1]$ i.e.,

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0\ :\ |t_0 - t_1| < \delta \implies d_\infty(\gamma(t_0), \gamma(t_1)) < \epsilon$$

Choose $\delta < \epsilon / d_\infty(f,g)$, then for any $s \in [a,b]$ we have

$$|t_0 f(s) + (1-t_0)g(s) - (t_1 f(s) + (1-t_1)g(s))| = |(t_0 - t_1)(f(s) - g(s))| \leq$$
$$\leq |t_0 - t_1||f(s) - g(s)|$$

And so taking the sup on both sides,

$$d_\infty(\gamma(t_0), \gamma(t_1)) \leq |t_0 - t_1| d_\infty(f,g) < \epsilon.$$

$C([a,b])$ is then path connected and so in particular connected.

Exercise 2. (1) Let $U \subseteq Y$ be an open subset. We need to check that the pre-image $f^{-1}(U)$ is open in X.

Since $f|_A\colon A \to Y$ is continuous, the pre-image $(f|_A)^{-1}(U)$ is open in A. Since A is open in X, this means that $(f|_A)^{-1}(U)$ is open in X. But we clearly see that $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$. So we have that $f^{-1}(U) \cap A$ is open in X.

In the same way as above we show that $f^{-1}(U) \cap B$ is open in X. Since $X = A \cup B$, we have $f^{-1}(U) = (f^{-1}(U) \cap A) \cup (f^{-1}(U) \cap B)$. This means that $f^{-1}(U)$ is the union of two open subsets of X, hence it is an open subset of X.

[Remark: the same proof works for an arbitrary open cover of X.]

(2) We want to show that $f\colon X \to Y$ is continuous by showing that the pre-image of every closed subset of Y is a closed subset of X (this is an equivalent condition for continuity!). The proof is very similar to (1). Let $Z \subseteq Y$ be an open subset.

Since $f|_A\colon A \to Y$ is continuous, the pre-image $(f|_A)^{-1}(Z)$ is closed in A. Since A is closed in X, this means that $(f|_A)^{-1}(Z) = f^{-1}(A) \cap Z$ is closed in X. Analogously $f^{-1}(B) \cap Z$ is closed in X.

Since $X = A \cup B$, we have $f^{-1}(Z) = (f^{-1}(Z) \cap A) \cup (f^{-1}(Z) \cap B)$. This means that $f^{-1}(Z)$ is the union of two closed subsets of X, hence it is a closed subset of X.

[Remark: the same proof works for any finite closed cover of X. Actually it works for any locally finite closed cover of X. Try to find a counterexample to the pasting lemma in the case of an infinite closed cover... see problem sheet 2]

Exercise 3. Since $\mathbb{R}^2$ is itself convex, it is enough to consider the case in which we work inside a convex subset $D \subseteq \mathbb{R}^n$. By a *convex subset* we mean a subset D of $\mathbb{R}^n$ such that, whenever we choose two points x and y in D, the segment between x and y is contained in D, i.e. for any $t \in [0,1]$ we have $(1-t)x + ty \in D$.

So let $\gamma_0\colon [0,1] \to D$ and $\gamma_1\colon [0,1] \to D$ be two paths with the same endpoints x_0 and x_1, i.e. $\gamma_0(0) = \gamma_1(0) = x_0$ and $\gamma_0(1) = \gamma_1(1) = x_1$. We want to construct a homotopy $H\colon [0,1] \times [0,1] \to D$ between these two paths. As suggested by the text we want that the family of paths $\gamma_t = H(\cdot, t)$, as $t \in [0,1]$, interpolates between γ_0 and γ_1, so for any $s \in [0,1]$ and $t \in [0,1]$ we want that $\gamma_t(s) = H(s,t)$ lies on the segment between $\gamma_0(s)$ and $\gamma_1(s)$, because we want to use the fact that D is convex. So let's consider the map $H\colon [0,1] \times [0,1] \to D$ defined by

$$H(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$$

for any $s, t \in [0,1]$. It is well-defined because D is convex. It is obvious that H satisfies the four properties we want. The only non-trivial thing is to show that H is continuous.

In order to show that H is continuous one can use the $\varepsilon \,\&\, \delta$ approach. Fix an arbitrary point $(\bar{s}, \bar{t}) \in [0,1] \times [0,1]$ and fix $\varepsilon > 0$. We want to show that there exists a neighbourhood U of $(\bar{s}, \bar{t})$ such that the image of U along H is contained in the ball of radius ε centred in $H(\bar{s}, \bar{t})$, i.e. $H(U) \subseteq B_\varepsilon(H(\bar{s}, \bar{t}))$. For any $i = 0, 1$, since γ_i is continuous in $\bar{s}$ we get that there exists $\delta_i > 0$ such that

$$|s - \bar{s}| < \delta_i \Longrightarrow \|\gamma_i(s) - \gamma_i(\bar{s})\| < \varepsilon/3.$$

We have

$$\begin{aligned} H(s,t) - H(\bar{s}, \bar{t}) &= H(s,t) - H(\bar{s}, t) + H(\bar{s}, t) - H(\bar{s}, \bar{t}) \\ &= (1-t)(\gamma_0(s) - \gamma_0(\bar{s})) + t(\gamma_1(s) - \gamma_1(\bar{s})) + (\bar{t} - t)(\gamma_0(\bar{s}) - \gamma_1(\bar{s})). \end{aligned}$$

For brevity set $C = \|\gamma_0(\bar{s}) - \gamma_1(\bar{s})\| \geq 0$ and $\delta = \min\{\delta_0, \delta_1\}$. We see that if $|s - \bar{s}| < \delta$ then

$$\|H(s,t) - H(\bar{s}, \bar{t})\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |t - \bar{t}|C.$$

Now set $0 < \delta^* \leq \varepsilon/(3C)$ (in the case $C = 0$ we allow δ^* to be any positive real number). It is clear that if $(s,t) \in [0,1] \times [0,1]$ is such that $|t - \bar{t}| < \delta^*$ and $|s - \bar{s}| < \delta$ then $\|H(s,t) - H(\bar{s}, \bar{t})\| < \varepsilon$. So we can take

$$U = \{(s,t) \in [0,1] \times [0,1] \mid |t - \bar{t}| < \delta^*, |s - \bar{s}| < \delta\}$$

which is clearly an open neighbourhood of $(\bar{s}, \bar{t})$ in $[0,1] \times [0,1]$.

[Remark: There is also a more conceptual way to show that H is continuous: this is based by observing that H is defined by sums, products and compositions of continuous functions. More precisely, we start by observing that the scalar multiplication $\cdot\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and the sum $+\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous (convince yourself about this!). Now consider the (obviously continuous) function $\varphi_0\colon \mathbb{R} \to \mathbb{R}$ defined by $\varphi_0(t) = 1 - t$ for any $t \in [0,1]$. Since $\gamma_0\colon [0,1] \to \mathbb{R}^n$ is continuous, by the properties of the product topology we know that the map $\gamma_0 \times \varphi_0\colon [0,1] \times [0,1] \to \mathbb{R}^n \times \mathbb{R}$, defined by $\gamma_0 \times \varphi_0\colon (s,t) \mapsto (\gamma_0(s), 1-t)$, is continuous. By composing $\gamma_0 \times \varphi_0$ with $\cdot$, we obtain that the map $\psi_0\colon [0,1] \times [0,1] \to \mathbb{R}^n$, defined by $\psi_0\colon (s,t) \mapsto (1-t)\gamma_0(s)$, is continuous. In a similar way one can show that the map $\psi_1\colon [0,1] \times [0,1] \to \mathbb{R}^n$, defined by $\psi_1\colon (s,t) \mapsto t\gamma_1(s)$, is continuous. By the properties of the product topology we have that the map $(\psi_0, \psi_1)\colon [0,1] \times [0,1] \to \mathbb{R}^n \times \mathbb{R}^n$, defined by $(\psi_0, \psi_1)\colon (s,t) \mapsto (\psi_0(s,t), \psi_1(s,t))$, is continuous. Now we compose (ψ_0, ψ_1) with $+$ and we get H.]

Exercise 4.

reflexivity: The constant homotopy is a homotopy from $\gamma_0(s)$ to itself, i.e., define

$$H(s,t) = \gamma_0(s) \text{ for all } s \in [0,1] \text{ and for all } t \in [0,1].$$

$H(s,t)$ it is obviously continuous and such that all the conditions are satisfied.

symmetry: Let $H(s,t)$ be an homotopy from $\gamma_0(s)$ to $\gamma_1(s)$ then we have an homotopy from $\gamma_1(s)$ to $\gamma_0(s)$ defined by

$$G(s,t) = H(s, 1-t).$$

This G we just defined is an homotopy, in fact

- $G(s,t)$ is continuous for all $(s,t) \in [0,1] \times [0,1]$: indeed $H(s,t)$ is continuous, $f(t) = 1-t$ in continuous and so is $H(s, f(t))$.
- $G(s,0) = H(s,1) = \gamma_1(s)$ for all $s \in [0,1]$;
- $G(s,1) = H(s,0) = \gamma_0(s)$ for all $s \in [0,1]$;
- $G(0,t) = H(0,1-t) = x_0$ for all $t \in [0,1]$;
- $G(1,t) = H(1,1-t) = x_1$ for all $t \in [0,1]$;

transitivity: Let $H^1(s,t)$ be the homotopy from $\gamma_0(s)$ to $\gamma_1(s)$ and $H^2(s,t)$ be the homotopy from $\gamma_1(s)$ to $\gamma_2(s)$, then we have an homotopy $G(s,t)$ from $\gamma_0(s)$ to $\gamma_2(s)$ defined by

$$G(s,t) = \begin{cases} H^1(s,2t) \text{ for all } s \in [0,1] \text{ and for } t \in [0,1/2] \\ H^2(s,2t-1) \text{ for all } s \in [0,1] \text{ and for } t \in [1/2,1] \end{cases}$$

This G we just defined is an homotopy, in fact

- $G(s,t)$ is continuous by Exercise 2.2 because $\{[0,1] \times [0,1/2], [0,1] \times [1/2,1]\}$ is a closed cover of the square $[0,1] \times [0,1]$;
- $G(s,0) = H^1(s,0) = \gamma_0(s)$ for all $s \in [0,1]$;
- $G(s,1) = H^2(s,1) = \gamma_2(s)$ for all $s \in [0,1]$;
- $G(0,t) = x_0$ for all $t \in [0,1]$; and $G(1,t) = x_1$ for all $t \in [0,1]$ it is as well clear from the definition of $G(s,t)$.