Solutions to Problem Sheet 7

Exercise 1. Let us consider the sequence $\{x_n \colon = 1/n\}_{n>0}$. For any n, $x_n \in (0,1)$, but 0, which is the limit of the sequence, is not in the open interval. Moreover, as the sequence converges to 0, any subsequence $\{x_{n_k}\}$ will converge again to 0. We then found a sequence $\{x_n\}_{n>0}$ is in $(0,1)$ which does not admit any subsequence which converges in $(0,1)$. Thus the open interval is not sequentially compact.

Exercise 2.

($\Rightarrow$) If X is compact, then any open cover has an finite open sub-cover. In particular, consider for $\epsilon > 0$ the open cover $\{B_\epsilon(x); x \in X\}$. Thus X is totally bounded.

Now, to show that X is complete we need to show that any Cauchy sequence is convergent. X compact $\iff$ X sequentially compact, so given $\{x_n\}_{n>0}$ there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ which converges to a certain $x \in X$. But in fact $\{x_n\}$ itself convergent to that $x \in X$. Indeed, fix $\epsilon > 0$, then for any $n, n_k > N(\epsilon)$ we have

$$d_X(x_n, x) \leq d_X(x_n, x_{n_k}) + d_X(x_{n_k}, x) < \epsilon/2 + \epsilon/2$$

where the first is the triangular inequality and the second follows from the fact that the sequence is Cauchy and the subsequence is convergent.

($\Leftarrow$) Conversely, assume X is totally bounded. **For metric spaces, sequential compactness is equivalent to compactness.** Thus we aim to show that X is sequentially compact. Let $(x_n)_{n\geq 0}$ be a sequence in X. We aim to show it has a convergent subsequence.

As X is totally bounded that there exists a finite set of indices, say $n_1, \ldots, n_k$ such that $X = \bigcup_{i=1}^k B_1(x_{n_i})$. By **pigeon-hole principle**, one of the balls must contain infinitely many terms of (x_n). This defines a subsequence of (x_n) which we call (x_{1n}). By definition (x_{1n}) is contained in a ball of radius 1.

We apply the same argument to balls of radius $1/2$ and (x_{1n}). This gives a subsequence (x_{2n}) contained in a ball of radius $1/2$. Continuing in the same fashion we obtain subsequences (x_{kn}) contained in balls of radius $1/k$.

Now, use a **diagonal argument**. For $k \geq 0$, define $x_k^* := x_{kk}$. We claim that $(x_k^*)_{k\geq 0}$ is a convergent subsequence. Indeed, let $\epsilon > 0$. Take N such that $1/N < \epsilon$ then for any $k \geq N$, all the x_k's are contained in a ball of radius $1/N < \epsilon$. So $(x_k^*)_{k\geq 0}$ is Cauchy, and hence convergent. Q.E.D.

Exercise 3. We begin by proving the following lemma.

Lemma. *Let X be a compact metric space and $V_1 \supseteq V_2 \supseteq \cdots$ be a sequence of nested closed subsets of X. Consider the intersection*

$$V_\infty = \bigcap_{n=1}^{\infty} V_n.$$

If $f \colon X \to \mathbb{R}$ is a continuous function then

$$\sup_{V_\infty} f = \inf \left\{ \sup_{V_n} f \,\middle|\, n \geq 1 \right\}. \tag{1}$$

Proof of the Lemma. If one of the V_n is empty, then V_∞ is empty and then $\sup_{V_\infty} f = -\infty$ and $\sup_{V_n} f = -\infty$. Thus the equality is obvious.

We may assume that V_n is non-empty for any n. By Exercise 3 in Problem Sheet 4 we know that V_∞ is non-empty. For any $n \geq 1$, we have that $V_\infty \subseteq V_n$ which implies that $\sup_{V_\infty} f \leq \sup_{V_n} f$; since this holds for any n we have shown the inequality $\leq$ in (1).

Now we want to prove the inequality $\geq$ in (1). Now let us fix $n \geq 1$. As V_n is closed in the compact topological space X, we know that V_n is compact. This implies that f attains maximum over V_n; in other words there exists $x_n \in V_n$ such that $f(x_n) = \sup_{V_n} f$. We have constructed the sequence $\{x_n\}_{n\geq 1}$. We notice that the sequence $\{f(x_n)\}_{n\geq 1}$ is (weakly) decreasing sequence of real numbers converging to the right hand side in (1).

As X is a compact metric space, it is sequentially compact. Therefore it is possible to extract a convergent subsequence $\{x_{n_k}\}_{k\geq 1}$ from the sequence $\{x_n\}_{n\geq 1}$. Let $\bar{x} \in X$ denote the limit of this convergent subsequence. We want to prove that $\bar{x}$ lies in V_∞. Let's fix an arbitrary $k^* \geq 1$; we know that for any $k \geq k^*$ the point x_{n_k} lies in $V_{n_{k^*}}$, so also the limit $\bar{x}$ lies in $V_{n_{k^*}}$ because $V_{n_{k^*}}$ is closed and hence contains all of its limit points. We have shown that $\bar{x}$ lies in $V_{n_{k^*}}$ for any $k^* \geq 1$. As $\{n_k\}_{k\geq 1}$ is an unbounded sequence of natural numbers, we have that

$$\bar{x} \in \bigcap_{k\geq 1} V_{n_k} = V_\infty.$$

As $x_{n_k} \to \bar{x}$ and f is continuous, the sequence of real numbers $\{f(x_{n_k})\}_{k\geq 1}$ converges to $f(\bar{x})$. But we knew that the limit had to be the right hand side in (1). This means that

$$\inf \left\{ \sup_{V_n} f \,\middle|\, n \geq 1 \right\} = f(\bar{x}) \leq \sup_{V_\infty} f.$$

And this shows ≥ 1 in (1). □

Now we go to the situation of the exercise. We have our compact metric space X with distance $d\colon X \times X \to \mathbb{R}$ and a sequence of nested closed subsets $V_1 \supseteq V_2 \supseteq \cdots$. One can show that d is a continuous function when $X \times X$ is equipped with the product topology by considering the following inequality, which is true for any $x_1, x_2, y_1, y_2 \in X$,

$$\begin{aligned}
|d(x_1,y_1) - d(x_2,y_2)| &= |d(x_1,y_1) - d(x_1,y_2) + d(x_1,y_2) - d(x_2,y_2)| \\
&\leq |d(x_1,y_1) - d(x_1,y_2)| + |d(x_1,y_2) - d(x_2,y_2)| \\
&\leq d(y_1,y_2) + d(x_1,x_2) \\
&= d_{X\times X}\left((x_1,y_1),(x_2,y_2)\right),
\end{aligned}$$

where $d_{X\times X}$ is the distance defined in Exercise 1 in the Extra Problem Sheet with $p = 1$.

Now one sees that

$$\operatorname{diam}(V_n) = \sup_{V_n \times V_n} d$$

for any n. By applying the lemma above to the function $d\colon X \times X \to \mathbb{R}$ and to the sequence $V_1 \times V_1 \supseteq V_2 \times V_2 \supseteq \cdots$ of closed subsets of the compact metric space $X \times X$ we get the required equality.

Exercise 4. We have seen in Problem Sheet 6, Exercise 3, point (1) that any two paths $f(t), g(t)$ in $\mathbb{R}^2$ with the same endpoints x_0, x_1 are homotopic, with the homotopy from $g(t)$ to $f(t)$ $H\colon [0,1] \times [0,1] \to \mathbb{R}^2$ given by

$$H(s,t) = s \cdot f(t) + (1-s) \cdot g(t).$$

In the case of the exercise we have

$$f(0) = g(0) = (1,0) \;\; f(1) = g(1) = (-1,0)$$

so the general fact we proved last week can be applied.

Exercise 5.

(1) Let x and y be two points in D. As D is star shaped, the straight segments $\gamma_1(t) = t \cdot x + (1-t) \cdot x_0$ and $\gamma_2(t) = t \cdot x_0 + (1-t) \cdot y$ are contained in D and they are obviously continuous paths connecting x to x_0 and x_0 to y. Then consider the path $\gamma\colon [0,1] \to D$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(2t) = (1-2t) \cdot x + 2t \cdot x_0 \text{ for t } \in [0,1/2] \\ \gamma_2(2t-1) = (2-2t) \cdot x_0 + (2t-1) \cdot y \text{ for t } \in [1/2,1] \end{cases}$$

this is clearly continuous and connects x and y. As they were arbitrary, this means D is path connected.

(2) Let $\gamma\colon [0,1] \to D$ be a path. We want to shrink γ to the constant path $\gamma_0\colon t \mapsto x_0$, as $t \in [0,1]$. Consider the homotopy $H\colon [0,1] \times [0,1] \to D$ defined by

$$H(t,s) = (1-s)\gamma(t) + sx_0$$

for any $s, t \in [0,1]$. One should prove that H is continuous in the same way as in Exercise 3 in Problem Sheet 6. It is clear that $H(\cdot, 0) = \gamma$, $H(\cdot, 1) = \gamma_0$.