Deformations of Gorenstein canonical toric singularities

Because we have time, I begin with a gentle introduction to deformations.

A Traditional Example (Pinkham)

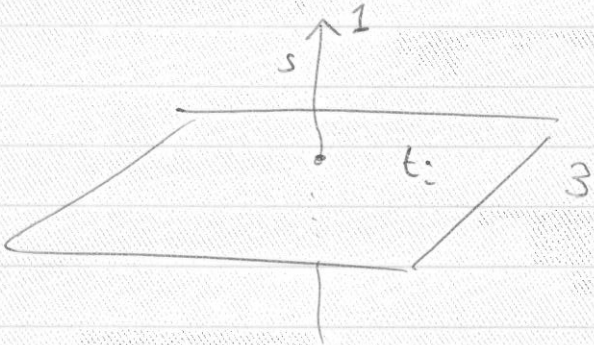
Cone over rational twisted quartic $\mathbb{P}^1 \hookrightarrow \mathbb{P}^4$

equations $\quad \operatorname{rk}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1$

a surface in $\mathbb{C}^5$

Versal deformation:

$$\operatorname{rk}\begin{pmatrix} y_0 & y_1+t_1 & y_2+t_2 & y_3+t_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1$$

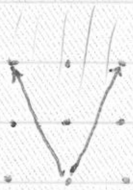


$$\operatorname{rk}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2+s & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \leq 1$$

TWO DIFFERENT PRESENTATIONS OF THE SAME IDEAL

How to see this in toric geometry?

$\sigma := \langle (\pm 1, 2) \rangle$



notational convention:

$(\cdots)$ in N

$[\cdots]$ in M

M = monomials = momentum polytope

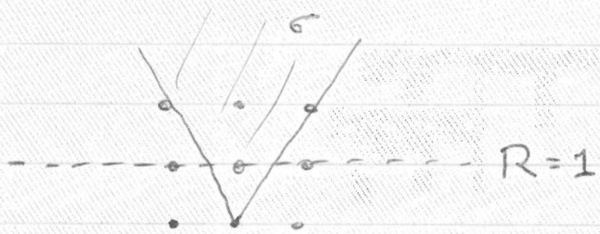
$N \ni$ fans

$\sigma^\vee = \langle [\pm 2, 1] \rangle$

<u>deformations</u>: first choose a degree $R \in M$ [b/c we are doing toric geometry, so everything is nicely multigraded]

and look for deformations in degree $-R \in M$

Easier case: assume R is primitive



Set $Q = \sigma \cap \{R = 1\}$

$= [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$

Now think about Minkowski sums:

$[-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, \frac{1}{3}] + [0, \frac{1}{6}]$

DON'T LIKE THIS b/c of lattice condition

if $Q = Q_0 + \cdots + Q_m$ (Minkowski sum of polyhedra) the <u>lattice condition</u> means:

- $\operatorname{tail} Q = \operatorname{tail} Q_i \ \forall i$

$\operatorname{tail} Q := \{ a \mid a + Q \subseteq Q \}$

nb $\underset{\text{not unique}}{Q = Q^{\text{compact}}} + \underset{\text{unique}}{\operatorname{tail}(Q)}$

nb Q compact $\Leftrightarrow \operatorname{tail}(Q) = \{0\}$

can't hold unless Q is a lattice polytope, so weaken as follows: (see overleaf)

- ~~$Q_0, \ldots, Q_m$ are lattice polytopes~~

i.e. not $\perp$ to faces

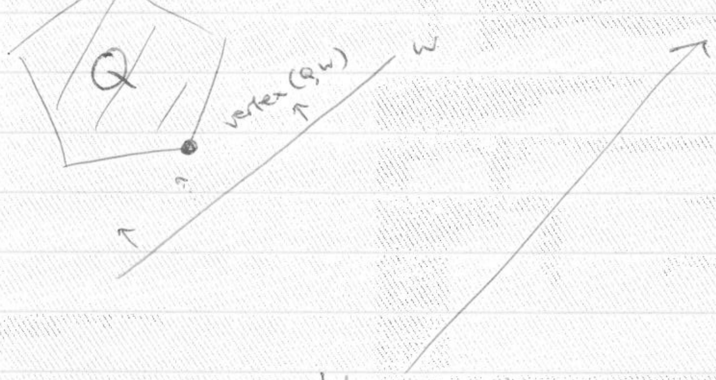
such that $w \in \mathrm{tail}(Q)^\vee$ <

— else $\nexists$ vertex(Q,w) b/c w is unbdd below on Q

- $\forall\ w \in M$ general, vertex$(Q, w) = \sum$ vertex(Q_i, w)

ALWAYS TRUE

and all but one of the vertices lie in N



this is the same as our deleted condition when Q is a lattice polytope, but different when Q is not a lattice point.

example: $[-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, \frac{1}{3}] + [0, 1/6]$ fails

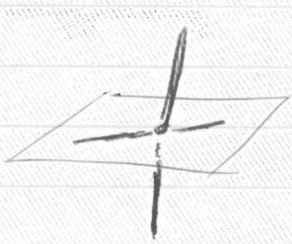
$= [-\frac{1}{2}, \frac{1}{2}] + \{0\}$ holds but ignore as trivial

$= [-\frac{1}{2}, 0] + [0, -\frac{1}{2}]$ holds

$= [-1, 0] + \{\frac{1}{2}\}$ holds

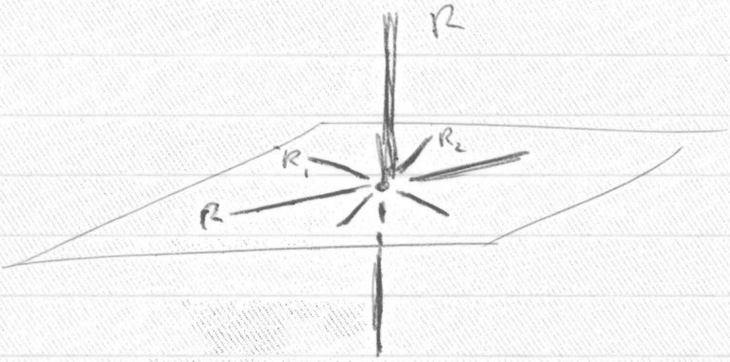
Claim: A splitting $Q = Q_0 + Q_1 + \dots + Q_m$ leads to an m-parameter family of deformations.

So far, have found

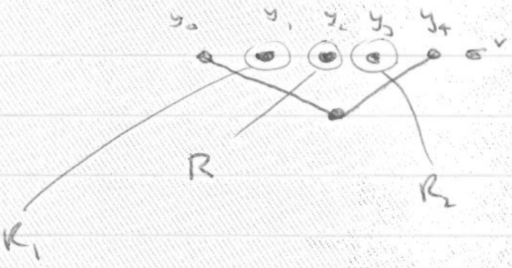


with $R = [0,1]$

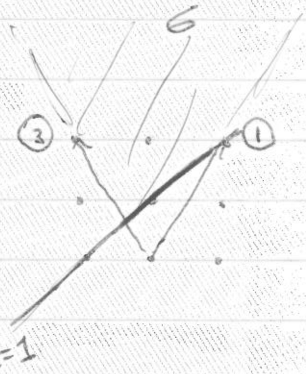
Nb homogeneous deformations do not give ~~all~~ deformations but rather a skeleton of (i.e. a basis for) the deformations:



what are the degrees?



do degree R_1 example: $R_1 = [-1, 1] \in M$



(r) = value of R_1

$$Q^1 = \sigma \cap \{R_1 = 1\}$$

$$= [-\tfrac{1}{3}, 1] \subseteq \mathbb{R}$$

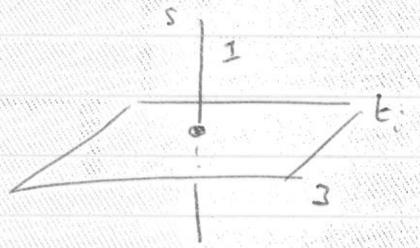
only splitting $[-\frac{1}{3}, 0] + [0, 1]$
which satisfies
the lattice condition

R	$[\frac{1}{2}, 0] + [0, \frac{1}{2}]$
	$[1, 0] + \{\frac{1}{2}\}$
R1	$[-\frac{1}{3}, 0] + [0, 1]$
R2	$[-1, 0] + [0, \frac{1}{3}]$

Which of these things is not like the others?

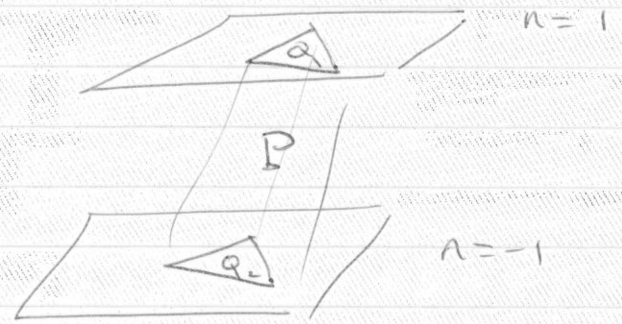
$[-1,0] + \{\tfrac{1}{2}\} \quad \longleftrightarrow \quad s$

others $\quad \longleftrightarrow \quad t_i$



How to construct the deformation?

dim $P = n$



call this $\tilde{N}$ $\quad$ nb $R^{\perp} \subset N$

$\underline{P} := \operatorname{conv}\{(Q_0, \underset{(1,0)}{e^0}), (Q_1, \underset{(0,1)}{e^1})\} \subseteq \underbrace{R^{\perp} \times \mathbb{Z}^{m+1}}_{\text{call this } \tilde{N}} \quad (m=1)$

$\bar{\sigma} := \operatorname{cone}(P) = (n+1)\text{-dimensional cone}$

nb $\tfrac{1}{2}Q \hookrightarrow P \quad (\text{height } \tfrac{1}{2})$

i.e. $\sigma \hookrightarrow \bar{\sigma}$ $\quad$ at least if $R \in \check{\sigma} \cap M$

in fact $\sigma \hookrightarrow \bar{\sigma} \xrightarrow[e_0, \dots, e_m]{} \mathbb{Q}^{m+1}_{\geq 0} = \mathbb{Q}^2_{\geq 0}$

Now apply the toric variety functor $TV(-)$.

$$
\begin{array}{ll}
TV(\sigma) & = \quad \text{original singularity } X \\
\big\downarrow \text{ closed embedding} & \\
TV(\tilde{\sigma}) & =: \quad \tilde{X} \\
\big\downarrow & \\
\mathbb{C}^{m+1} &
\end{array}
$$

b/c of $TV(\mathbb{Q}^{m+1}_{\geq 0}) = \mathbb{C}^{m+1}$

also $\mathbb{Q}_{\geq 0} \cdot \overbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}^{\text{call this } \mathbb{1}} \hookrightarrow \mathbb{Q}^{m+1}_{\geq 0}$

and σ = preimage of $\mathbb{Q}_{\geq 0} \mathbb{1}$

so get $\mathbb{C} . \mathbb{1} \hookrightarrow \mathbb{C}^{m+1}$ and:

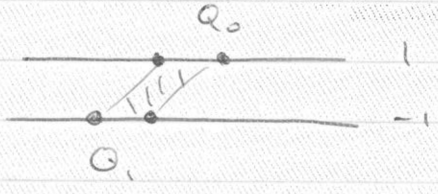
$$
\begin{tikzcd}
X = TV(\sigma) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{X} = TV(\tilde{\sigma}) \arrow[d, "\text{flat}"] \\
\{0\} \arrow[r, hook] & \mathbb{C}^m
\end{tikzcd}
$$

where

$$
\begin{array}{ccccccc}
TV(\sigma) & \hookrightarrow & TV(\tilde{\sigma}) & \longrightarrow & \mathbb{C}^{m+1} & \longrightarrow & \mathbb{C}^{m+1}/\mathbb{C}.\mathbb{1} \\
\| & & \| & & \uparrow & & \cong \\
X & & \tilde{X} & & \mathbb{C}.\mathbb{1} & & \mathbb{C}^m
\end{array}
$$

e.g. :

think



but actually use the line $x+y=1$ in $\mathbb{Z}^2$ as before ~~as be~~

$P = \mathrm{conv}(\text{ these })$

$\bar{\sigma} = \mathrm{cone}\left((0,1,0),\ (\tfrac{1}{2},1,0),\ (-\tfrac{1}{2},0,1),\ (0,0,1) \right)$

$\bar{\sigma} = \langle (0,1,0),\ (1,2,0),\ (-1,0,2),\ (0,0,1) \rangle$

and $\quad \bar{\sigma} \xrightarrow[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{} \mathbb{Q}^2_{\geq}$ $\quad$ is project to the last 2 co-ordinates

what is the fiber over the diagonal?

$$\begin{array}{ccc} N & \longrightarrow & \tilde{N} \\ \| & & \| \\ \mathbb{Z} \text{ or } \mathbb{Q} & a \longmapsto \begin{pmatrix} a \\ a_2 \\ a_2 \end{pmatrix} & \mathbb{Q}^3 \end{array}$$

$\tilde{N}$
 $\cup$

check $\quad \sigma = \langle (\pm 1, 2) \rangle \longmapsto \langle (\pm 1, 2, 2) \rangle$

which really is the fiber over the diagonal

~~$\sigma \hookrightarrow \tilde{\sigma}$, $\downarrow$, $\{0\} \hookrightarrow \mathbb{Q}^{m+1}_{\geq 0} / \mathbb{Q}.1$~~

so:
$$\begin{tikzcd} \sigma \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\sigma} \arrow[d] \\ \mathbb{Q}_{\geq 0} 1 \arrow[r, hook] & \mathbb{Q}^{m+1}_{\geq 0} \end{tikzcd}$$

$\therefore$ applying TV gives

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \tilde{X} \arrow[d, "\pi"] \\ \mathbb{C} \arrow[r, hook, "\mathbb{1}"] & \mathbb{C}^{n+1} \end{tikzcd}$$

$f = x^R$

now divide out by $\mathbb{C}$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{X} \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C}^n = \mathbb{C}^{n+1}/\mathbb{C}\mathbb{1} \end{tikzcd}$$

<u>So far</u>: we have seen

1. decent deformations when $R \in \sigma^\vee \cap M$ (i.e. global, not just over a fat point)

2. versal deformations

<u>Very roughly</u>: There is a T^1 space

$T^1(-R) \sim V(Q)/1$

" $\mathrm{Def}_X(\mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle)_{-R}$ — graded ~~isotypic~~ piece

where $V(Q)$ is a vector space of all

generalized Minkowski summands of Q

$$\| $$

$$\left(t \in \mathbb{Q}^{\text{compact edges of } Q} \;\middle|\; \sum_i \varepsilon_i t_i d^i = 0 \right)$$

call the compact edges $d^i \in N_Q$
and the 2-faces $\varepsilon = \{\pm 1, 0\}^{\text{edges}}$

"generalized Minkowski sum" b/c edges can be negative

so $\sum_i \varepsilon_i d^i = 0$

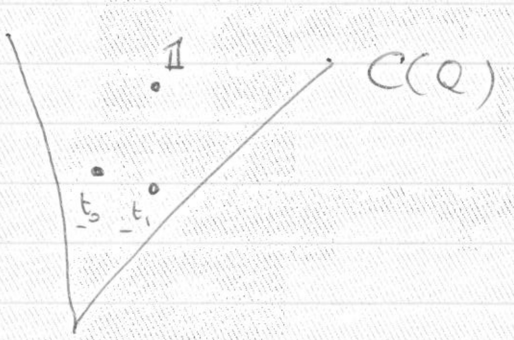
now $\mathcal{V}(Q) \supset C(Q) := \mathcal{V}(Q) \cap \mathbb{Q}^{\text{edges}}_{\geq 0}$

"actual Minkowski sums"

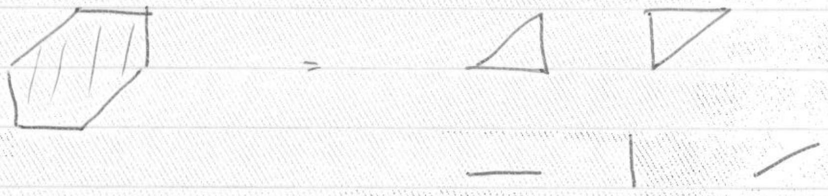
Now $Q = Q_0 + Q_1$

$$\Updownarrow$$

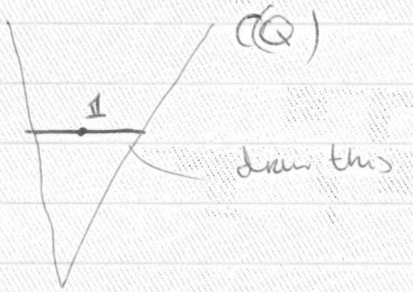
$\underline{1} = \underline{t}_0 + \underline{t}_1$ where all $t_i^j \geq 0$
i.e. $\underline{t}_0, \underline{t}_1 \in C(Q)$



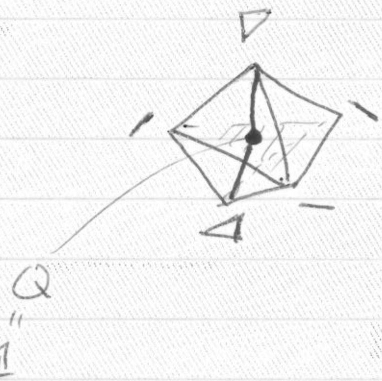
example

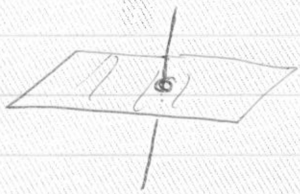


$C(Q) = 4$-dimensional cone $\left(\begin{array}{l}6 \text{ edges} \\ 1 \text{ 2-dim'l constraint}\end{array}\right)$



picture it via a cross-cut





versal def.
has 2d piece
coming from

/ + — + \

and 1d piece
from
▽ + △

This works out exactly in nice cases.

Assume: $\sigma = \text{Cone}(P)$ is Gorenstein

$\qquad\qquad\qquad\;\,\|$

lattice polytope with primitive edges

anticanonical divisor

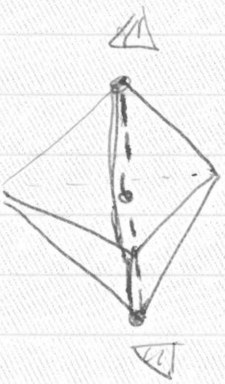
$$Q = \sigma_A\{R^* = 1\} \qquad (\rightsquigarrow T' = T'(-K^*))$$

Define

$$\begin{array}{ccc} \tilde{C}(Q) & \longrightarrow & C(Q) \\ \cap & & \cap \\ N_Q \times V_Q & & V(Q) \end{array}$$

= Moduli space of Minkowski summands

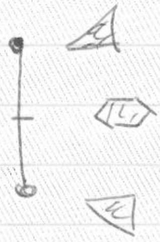
$\tilde{C}(Q)$ = universal Minkowski summand



$C(Q) \cong \mathbb{Q}^2_{\geq 0}$ 2-dim'l cone from factorization

now that 2-d cone = cone over edge

and so the universal family we have



i.e. our construction from previously

$$\begin{tikzcd} \tilde{C}(Q) \arrow[r] & C(Q) \arrow[r] & \mathbb{Q}^{\text{edges}}_{\geq 0} \\ \sigma \arrow[u, hook] \arrow[r, leftrightarrow] & 1 \cdot \mathbb{Q}_{\geq 0} \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

$\supseteq \mathbb{Q}^2_{\geq 0}$ for us

b/c

2 edges

Now apply $TV(-)$

$$\begin{tikzcd} \tilde{X} \arrow[r] & S \arrow[r] & \mathbb{C}^{\text{edges}} \\ X \arrow[u, hook] \arrow[r] & \mathbb{C}\mathbb{1} \arrow[u] \arrow[ur, hook, "\text{diagonal}"'] & \end{tikzcd}$$

now pretend $S \to \mathbb{C}^{\text{edges}}$ is an embedding

(is this true? maybe not but v. difficult to find a counterexample)

and divide out

$$\begin{tikzcd} \tilde{X} \arrow[r] & \mathbb{C}^m := \mathbb{C}^{\text{edges}}/\mathbb{C}\cdot\mathbb{1} \\ X \arrow[u, hook] \arrow[r] & pt \arrow[u, hook] \end{tikzcd}$$

but this is not yet a def diagram b/c $\tilde{X} \to \mathbb{C}^m$ is not flat; if you try to prove it you show that you can always lift syzygies etc. provided you had equations of S (which you don't b/c you forgot them)

$$\mathbb{C}^m = \mathbb{C}^{\text{edges}} / \mathbb{C}\mathbb{1} = T^1\text{-space}$$

| has infinitesimal deformations |

so if $\mathbb{C}^{\text{edges}} \xrightarrow{e} \mathbb{C}^{\text{edges}}/\mathbb{C}\mathbb{1}$

and we want a parameter space

$$\bar{M} \subseteq \mathbb{C}^{\text{edges}}/\mathbb{C}\mathbb{1}$$

Seek $e^{-1}(\bar{M}) \subseteq S'$

i.e. define $\bar{M}$ = maximal s.t. $e^{-1}(m) \subseteq S'$ $\forall\, m \in \bar{M}$

<u>equations</u> of $\bar{M}$ are easy to find too. They are

$$\left\{ \underline{t} \in \mathbb{C}^{\text{compact edges}} \;\middle/\; \sum_i \varepsilon_i t_i^k d_i = 0 \text{ for all } k \geq 1 \right\}$$

$\bar{M} \subseteq T^1$

↑ true versal base space

↑ infinitesimal def. space

nb $k = 1$ defined T^1!

The k you need to take is bdd by something like the second-smallest width of the polytope

equations come from $T^1 \cup T^1 \to T^2$ have cup product

etc.

<u>Bitter truth</u> this only holds in isolated Gorenstein cases which are rigid anyway in dim ≥ 3

Gorenstein $\Rightarrow$ deformations only live in $-R^*$ degree; in general have many degrees

<u>What if $R \notin \sigma^\vee \cap M$</u>

Consider σ with $X = TV(\sigma)$

$Q = \sigma \cap \{R = 1\}$ nb now cannot reconstruct σ from Q

now set $\tau = \sigma \cap \{R \geq 0\}$ $(\Rightarrow R \in \tau^\vee \cap M)$

so <u>can</u> deform τ

$$\begin{array}{ccc} X_\tau & \hookrightarrow & \tilde{X}_\tau \\ \downarrow & & \downarrow \text{flat} \\ \{0\} & \hookrightarrow & \mathbb{C}^m \end{array}$$

and $\tau \subseteq \sigma \Rightarrow \exists\ X_\tau \to X_\sigma$ blow-up map

$$\begin{tikzcd} TV(\tau) \arrow[r] \arrow[d, "\text{blow-up}"'] & TV(\tilde{\tau}) \arrow[r, "\text{flat}"] \arrow[d] \arrow[dd, bend left] & \mathbb{C}^m \\ TV(\sigma) \arrow[r] \arrow[dr] & \tilde{X} \arrow[ur] \arrow[d] & \\ & TV(\tilde{\sigma}) \arrow[uur, dashed, "\text{rational map}"'] & \end{tikzcd}$$

$\tilde{\sigma} := \cancel{}\, \tilde{\tau} + \sigma$

where $\tilde{X}$ resolves the indeterminacy of the rational map

now $\tilde{\sigma} = \tilde{\tau} + \tilde{\tilde{\sigma}}$

$\Downarrow$

$\tilde{\sigma}^\vee = \tilde{\tau}^\vee \cap \tilde{\tilde{\sigma}}^\vee$

$\tilde{\sigma}^\vee \subset \tilde{\tau}^\vee$

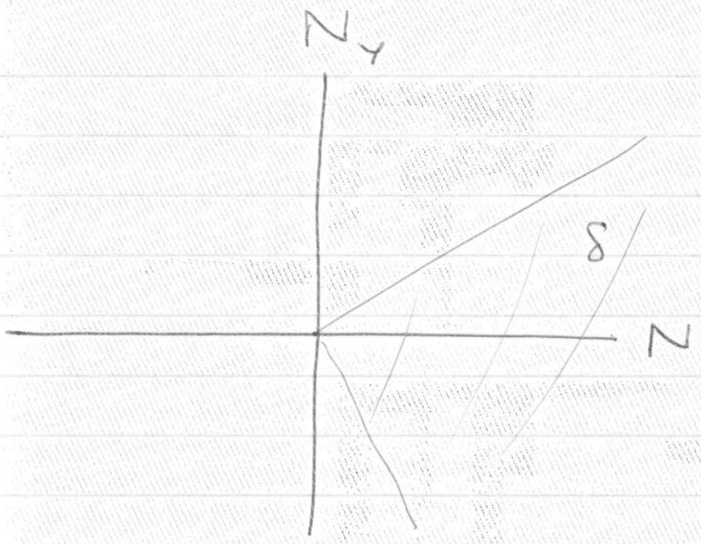
$x^{R_0} x^{R_1} \longmapsto x^R$

b/c diagonal was $x^{R_0} - x^{R_1}$ or whatever

$$\tilde{X} = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \underline{M}][x^{R_0} - x^{R_1}]$$

not toric any more

Will give a new point of view which includes all constructions we have discussed so far in a very natural way.



$\mathbb{Z}^n = N \oplus N_Y$

δ = cone

have

$0 \to N \to \mathbb{Z}^n \xrightarrow{p} N_Y \to 0$

so project onto N_Y:

$p(\delta) = N_Y$ but, less naively,

$\Sigma :=$ image fan of δ

$\exists!$ fan which is the coarsest possible refinement of $\{p(\tau) : \tau \text{ a face of } \delta\}$; this is the def^n of Σ.

Set $Y = TV(\Sigma)$ $(= \mathbb{P}^1)$

goal is to forget part of the torus action

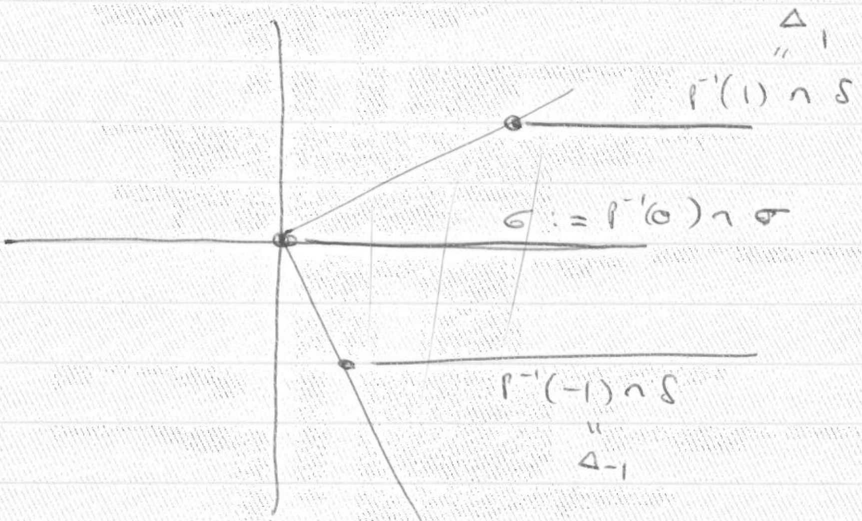
$0 \to N \to \mathbb{Z}^n \longrightarrow N_Y \to 0$

sublattice ⇝ subtorus

Want to understand our varieties as invariant w.r.t N not $\mathbb{Z}^n$

torus of   torus of

Can't recover everything from Σ b/c lost information, need also to know



- $\Sigma = p(\sigma)$
- $Y = TV(\Sigma)$
- $\sigma, \Delta_{-1}, \Delta_1$ —— $\Delta_i \in N_{\mathbb{Q}}$ polyhedron s.t. tail cone $= \sigma$

So $Y, \sigma, \Delta_{-1}, \Delta_1 \longleftrightarrow \delta$

reinterpret:

$\mathbb{P}^1$, $\overline{orb_{\mathbb{Z}}(1)}$ = divisor in $\mathbb{P}^1$ $\{0\}$

$\overline{orb(-1)}$ = divisor in $\mathbb{P}^1$ $\{\infty\}$

Set $\mathcal{D} = \Delta_1 \times \{0\} + \Delta_{-1} \times \{\infty\}$

or, relabelling, $\mathcal{D} = \Delta_0 \times \{0\} + \Delta_\infty \times \{\infty\}$

So, new structure:

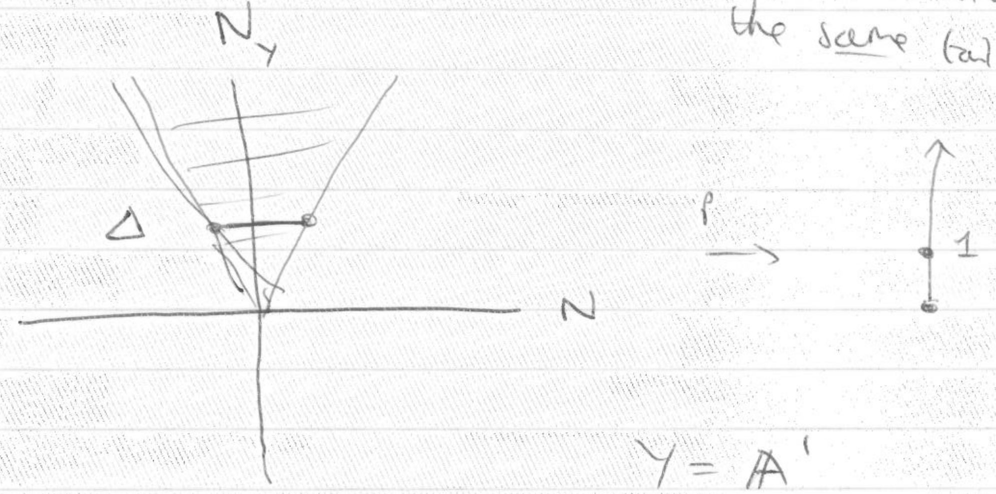
$Y =$ quasiprojective variety (or scheme)

$\mathcal{D} = \sum_i \Delta_i \otimes D_i$

$D_i \subseteq Y$ prime divisor

$\Delta_i \subseteq N_{\mathbb{Q}}$ polyhedron with the same tail cone.

<u>example</u>:



$Y = \mathbb{A}^1$

$\mathcal{D} = \Delta \otimes \{0\}$

or, pretend we really got $\mathbb{P}^1$ and:

$Y = \mathbb{P}^1 \qquad \mathcal{D} = \Delta \times \{0\} + \emptyset \times \{\infty\}$

So we can always pretend Y is projective.

Now do the same thing as we did before, two hours ago.

(ie on prev page)

NOTATION SWITCH $\qquad \mathbb{Z}^n$ before $= N$ now

Take σ, $R \in M$

$$0 \to R^\perp \longrightarrow N \xrightarrow{R} \mathbb{Z} \to 0$$

we are deforming now <u>in degree zero</u> so the guy action (for the smaller torus $\mathrm{Torus}(R^\perp)$) survives to nearby fibres

Then $\quad X = TV(\sigma) \rightsquigarrow Y = \mathbb{P}^1$

$\mathcal{D} = (\sigma \cap \{R=1\})\,\{0\}$
$\quad Q$
$\quad + (\sigma \cap \{R=-1\})\,\{\infty\}$

Look! this is what we had before!

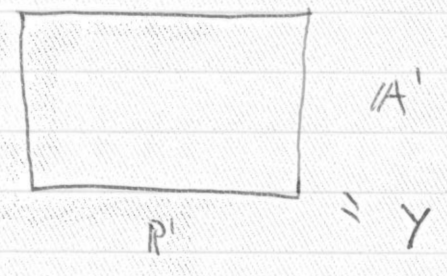
Now to deform the whole guy uses splitting $Q = Q_0 + Q_1$

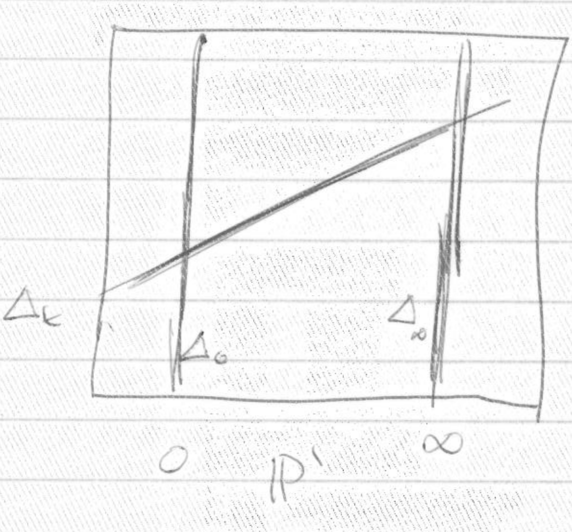
Imagine moving the divisors a little bit: should get flat family. Should hope: divisors coming together $\rightsquigarrow$ Minkowski sum.

$$\mathcal{D}_t := Q^0 \times \{0\} + Q^1 \times \{t\} + (\sigma \cap \{R=-1\}) \times \{\infty\}$$

where $Q = Q^0 + Q^1$

now





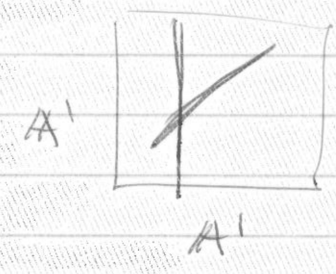
$\mathbb{A}^1$

$$\Delta_t = Q^0 \times \Delta_0 + Q^1 \times \Delta_t + [\{R = -1\} \cap \sigma] \times \Delta_\infty$$

NB if $R > 0$ then this disappears

$\Rightarrow$ get

like the $\phi \times \{\infty\}$ example before



$\mathbb{A}^1 \times \mathbb{A}^1$ with toric divisors

$\Rightarrow$ total space is toric

So when is this ~~total space~~ nearby fiber smooth? (Ilten–Vollmert)

given such $(Y, \mathcal{D})$, how do I obtain info. about the corresponding $\mathcal{X}$?

Given $u \in \mathrm{Tail}(\mathcal{D})^\vee$ consider

$$\min \langle \Delta_i , u \rangle$$

$\rightsquigarrow$ get $\mathcal{D}(u) = \sum_i (\min \langle \Delta_i , u \rangle) D_i$

which is a $\mathbb{Q}$-divisor on Y

so form $A = \bigoplus_u \mathcal{O}_Y(\mathcal{D}(u))$

$$\begin{tikzcd}
\tilde{X} = \operatorname{Spec} A \arrow[d] \arrow[r] & \operatorname{Spec} \Gamma(Y, A) = X \arrow[d] \\
Y \arrow[r] & Y_0 = \operatorname{Spec} \Gamma(Y, \mathcal{O}_Y)
\end{tikzcd}$$

relative spectrum

(if Y was affine, $Y_0 = Y$; else something gets contracted but we can understand it)

First try to understand singularities of $\tilde{X}$

this is local on Y (assume Y a curve) $\rightsquigarrow$ have toric models locally