Toric deformations of some spherical Fano varieties

Connection to the workshop:

toric mirror symmetry
(either for CY complete intersections in toric varieties, or toric varieties themselves)

CY in toric $\overset{\text{mirror}}{\rightsquigarrow}$ CY in toric

but there are some examples:

replace toric varieties by { Gr(k, n)
Flag variety

$\overset{\text{mirror}}{\rightsquigarrow}$ very special families of CY in toric

More general

spherical varieties $\rightsquigarrow$?

$\subset$ some toric varieties

$\cup$ Flag varieties

$\cup$ grassmannians

Want to construct mirrors for Fano varieties
Spherical varieties are a good class to consider

<u>Review</u>: spherical methods, and some instructive examples

of toric surfaces are an extremely instructive class of toric varieties
we will see a similarly-instructive set of spherical varieties: the
$SL(2)$-embeddings

$SL(2)$ is $xy - zt = 1$; embed $\mathcal{U} \cong SL(2) \hookrightarrow X$

<u>Def^n</u>: Let G be a reductive group and X a G-variety. X is spheric if $B \subset G$ has open dense orbit in X

Then X is a (partial?) compactification of G/H where H is the stabilizer of a point in the open B-orbit on G.

For us
$$G = SL(2) \times \mathbb{C}^\times$$
$$X = SL(2)$$

X has a left-$SL(2)$-action
and a right-$\mathbb{C}^\times$-action
where $\mathbb{C}^\times = T < SL(2)$ maximal torus

$\Rightarrow$ $G = SL(2) \times \mathbb{C}^\times$ acts on X.

X is G-spherical. $B = \begin{pmatrix} a & * \\ 0 & 1/a \end{pmatrix} \times \mathbb{C}^\times$

$\begin{pmatrix} a & * \\ 0 & 1/a \end{pmatrix} \Big\backslash X \cong \mathbb{P}^1$ and $\mathbb{C}^\times \circlearrowright \mathbb{P}^1$ with dense orbit

Open orbit $\mathcal{U} \cong G/H$ where $\dim H = 1$
$$\begin{array}{c} \mathcal{U} \\ \| \\ X \end{array}$$

<u>Toric compactification</u>: $K[T] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$

so embedding $T \hookrightarrow X$ s.t. $T \circlearrowright X$

can look for toric divisors $\rightsquigarrow$ valuations on $\mathbb{C}(T)$

$v(xy) = v(x) + v(y)$

$v(x+y) \geq \min(v(x), v(y))$

decompose $K[T] = \bigoplus_{\substack{\chi \in X(T) \\ \text{characters}}} \mathbb{C}[t^{\chi}]$

Any equivariant valuation is uniquely defined by its values on a basis for $X(T)$

$\rightsquigarrow$ gives an element of $X(T)^{\vee}$

and now can look at combinatorial structure

<u>Spherical compactification</u> has many similar ideas

<u>example</u> $\mathbb{C}[SL(2)] = \bigoplus_{n \geq 0} V_n \otimes V_n^*$

where V_n = standard rep^{n} of dimension $n+1$
= polynomials of degree n

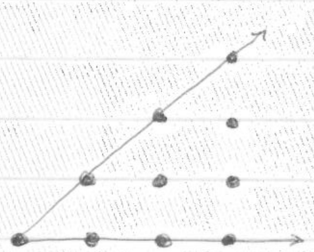
has L and R action

$$= \bigoplus_{n \geq 0} \left(V_n \right)^{\dim V_n}$$ (using L action only)

[different?]

The extra $\mathbb{C}^\times$ splits this $V_n \otimes V_n^*$ into 1-dim'l irreps:

$$\mathbb{C}[SL(2)]^{\mathbb{C}^\times}_{SL_2} = \bigoplus_{\substack{n \geq 0 \\ 0 \leq m \leq n+1}} V_{n,m} \qquad (\star)$$



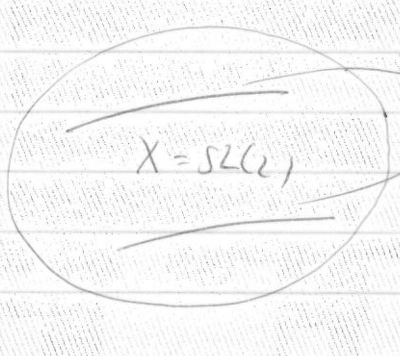
Denote by $B_u \subset B$ unipotent subgrp

$$\mathbb{C}[SL(2)]^{B_u} \simeq \mathbb{C}[x_1, x_2] \quad \text{b/c} \quad SL(2)/B_u \simeq \mathbb{C}^2 \setminus \{0\}$$

and the (n,m) in $(\star)$ can be identified as monomials in $\mathbb{C}[x_1, x_2]$.

(b/c reductive group irreps have highest weight vectors which are e'vals for the unipotent subgrp)

$B \subset G$



2-dim'l orbits $(x_1 = 0)$ $(x_2 = 0)$

Call the two distinguished orbits in $SL(2)$ colours.

Proceed as before, looking for valuations of the function field

$\uparrow$

valuations of co-ordinate ring

This is determined by $v(x_1)$, $v(x_2)$

in toric case there could be arbitrary

in spherical case this is not so.

$$\mathbb{C}[SL(2)] = \mathbb{C} \oplus \underbrace{V_1 \otimes V_1^*}_{\substack{\text{4-dim}^{l}\text{ rep}^{n} \\ \text{of } SL_2}} \oplus \cdots$$

$\mathbb{C}^*$-action on V_1^* splits us into cpts

2 highest wt vectors x_1, x_2

$$\begin{array}{|c|}\hline x_1 \\ x_3 \\ \hline\end{array} \quad \begin{array}{|c|}\hline x_2 \\ x_4 \\ \hline\end{array}$$

two std reps of $SL(2)$

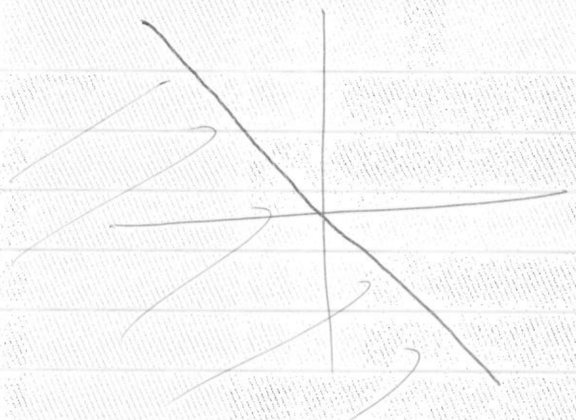
$v(x_1) = v(x_3)$ and $v(x_2) = v(x_4)$ b/c we look for invariant valuations

But $x_1 x_4 - x_2 x_3 = 1$ (b/c SL_2)

and $v(1) = 0 \Rightarrow v(x_1 x_4 - x_2 x_3) = 0$

~~$\Rightarrow v(x_1 x_4) + v(x_2 x_3) \geq 0$~~

$\Rightarrow a + b \leq 0$ where $a = v(x_1)$, $b = v(x_2)$



$a+b \leq 0$ is the valuation cone

NTS this is actually true not just $a+b \leq 0 \supseteq$ valuation cone

by showing that $\exists$ a compactification realising any such a, b

Consider $\mathbb{C}^5 = \underset{V_1}{\overset{x_1\, x_2}{\mathbb{C}^2}} \oplus \underset{V_1}{\overset{x_3\, x_4}{\mathbb{C}^2}} \oplus \underset{V_0}{\overset{x_0}{\mathbb{C}}}$ as $SL(2)$-modules

and $Y_{a,b} \subset \mathbb{C}^5$ defined by

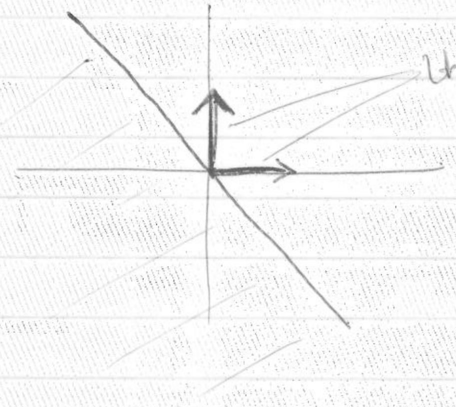
$$x_0^{-a-b} = x_1 x_4 - x_2 x_3$$

This is 4-dimensional; an affine variety.
Consider the $\mathbb{C}^\times$-action on $Y_{a,b}$ given by weights

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_0 \\ t^{-a} & t^{-b} & t^{-a} & t^{-b} & t \end{array}$$

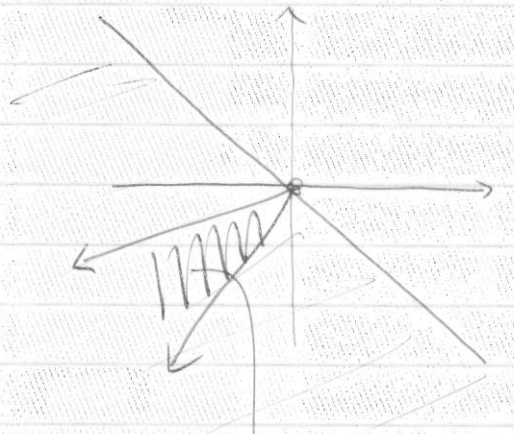
and take $Y_{a,b} /\!\!/ \mathbb{C}^\times$

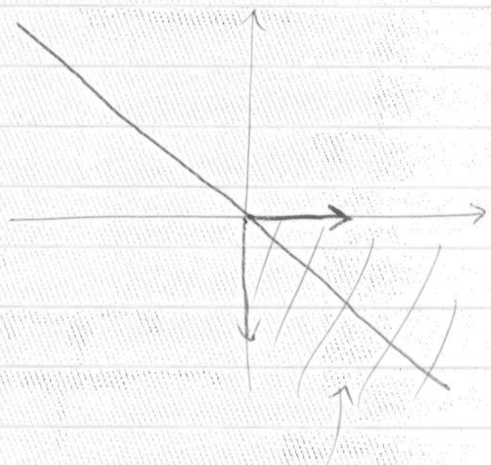
This has associated valuation $v(x_1) = a$
$v(x_2) = b$

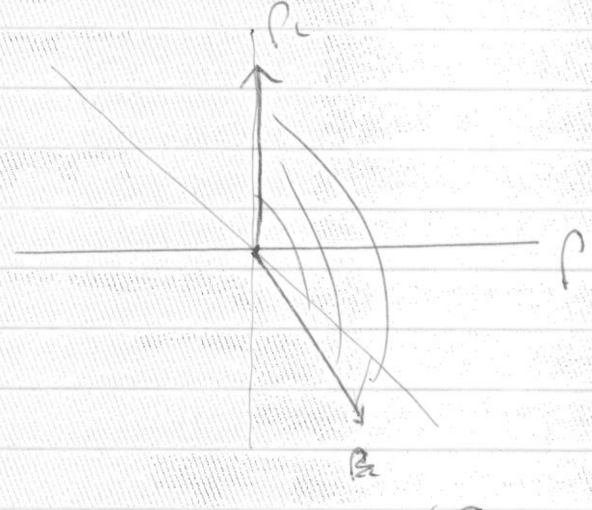


these are B-invariant divisors not G-invariant divisors

What happens in codim ≥ 2?   Fans etc, as follows.







"coloured cone" $\rightsquigarrow$ $(C, \{\rho_2\})$

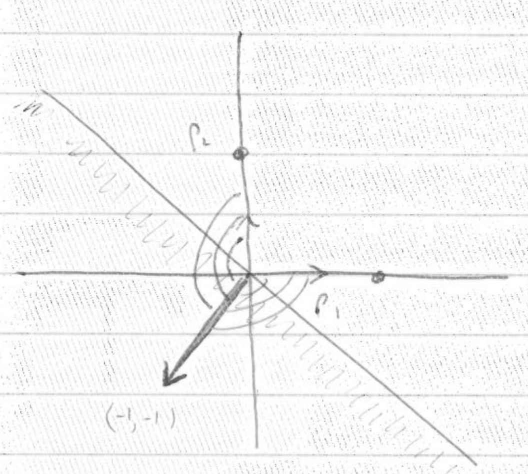
$\searrow$ $(C, \{\rho_1, \rho_2\})$

<u>different</u> as coloured cones

Cones such as this define <u>simple</u> spherical varieties.

These are always quasiprojective but not necessarily affine.

<u>example</u> $x_1 x_3 - x_2 x_4 = x_0^2 \subset \mathbb{P}^4$



$(-1,-1)$ is the divisor $x_0 = 0$

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

~~the~~ closed orbits $x_1 = x_3 = 0$

$x_2 = x_4 = 0$

2 cones

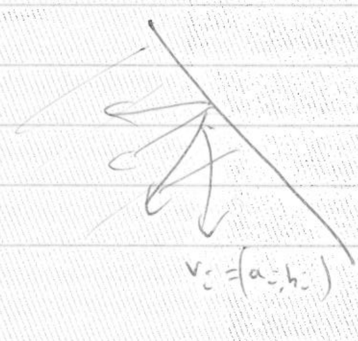
Now toric varieties are $X = \mathbb{C}^N /\!/ (\mathbb{C}^\times)^r$

$N = \#$ rays

$r =$ Picard rank

e.g. $\mathbb{P}^n = \mathbb{C}^{n+1} /\!/ \mathbb{C}^\times$

$(SL(2) \times \mathbb{C}^\times)$-spherical varieties are quotient hypersurfaces



vectors $v_1, \dots, v_r$

take $\mathbb{C}^4 \oplus \mathbb{C}^r$

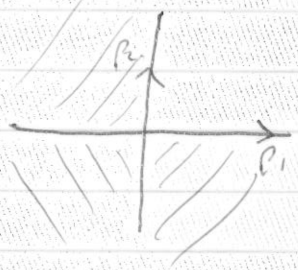
$\cup$

$x_1 x_4 - x_2 x_3 = y_1^{-a_1 - b_1} \cdots y_r^{-a_r - b_r}$

Now take $(\mathbb{C}^\times)^r$ - action of appropriate weight and can get X as quotient (Fatimah - Batyrev's student)

smooth (in this case) $\iff$ rays of cones all form lattice bases

<u>example 2</u>



$X = SL(3)/B \subset \mathbb{P}^2 \times \mathbb{P}^2$

$x_1 x_4 - x_2 x_3 = y_1 y_2$

how to find toric degenerations of this?

<u>example 3</u>

$\mathbb{P}^2 \times \mathbb{P}^1$
$\cup$
$SL(2)$

equivariant compactification

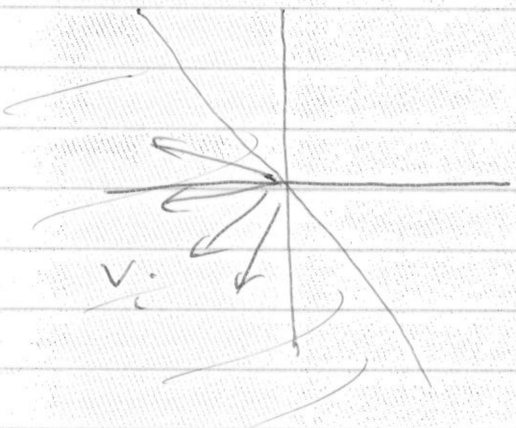
sometimes spherical varieties are toric

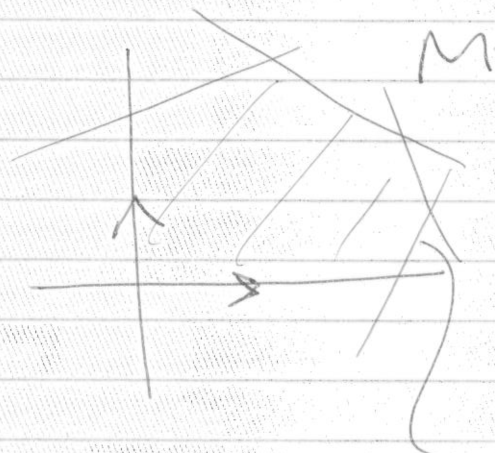
To speak about toric degeneration need to talk about moment polytopes etc.

$$\sum_{i=1}^{i=r} d_i D_i \qquad D_i \longleftrightarrow v_i = (a_i, b_i)$$

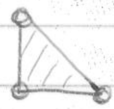
$a_i + b_i \leq 0$

N





$\langle v_i, * \rangle \geq -d_i$

e.g. quadric  = 

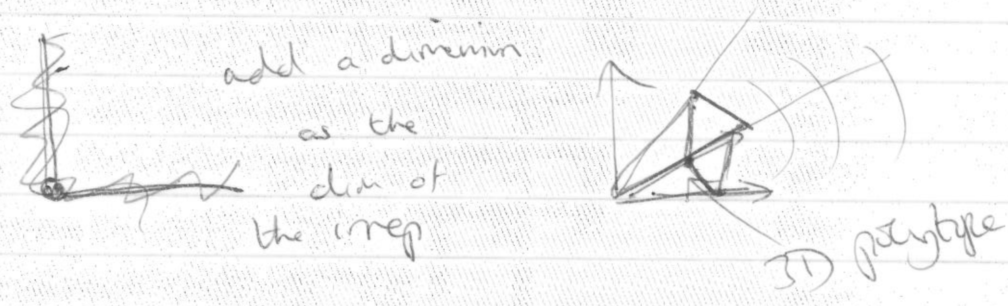
$1D_0 + 1D_1 + 1D_2$ std simplex

but $x_0^2 = x_1 x_3 - x_2 x_4 \subset \mathbb{P}^4$ and $\mathcal{O}(1)$ on $\mathbb{P}^4$ has 5-dim'l rep. so where is this

why not 5 lattice points in moment polytope

but recall we only see highest weights, so:

e.g. put height over this



Altmann

This is the "global Okounkov body"

and take that toric variety, there is a degeneration to this toric variety

i.e. the toric variety with this moment polytope

In our example

$$x_0^2 = x_1 x_4 - x_2 x_3$$

$$\Big\}$$

$$0 = x_1 x_4 - x_2 x_3$$

singular quadric
"
toric var.

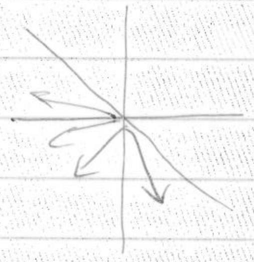
NB even if you start with a smooth toric variety this ~~deforms guy can be~~ degeneration can be to a singular toric variety

ALWAYS the singularity is at only one point

Vasily: What about Grassmannians?

Batyrev: [... omitted ...]

example



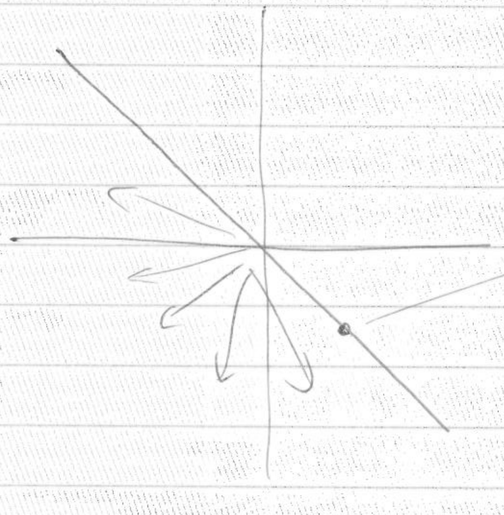
$\rightsquigarrow$ spherical variety which is a compactification of SL_2

What are the divisors here?

They are $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-a_i) \oplus \mathcal{O}(-b_i)) = \mathbb{F}_{(a_i - b_i)}$

Track the boundary divisors under the
$$\begin{array}{c} SL(2) \\ \cap \\ \mathbb{P}^2 \times \mathbb{P}^1 \\ \downarrow \\ \text{singular toric} \end{array}$$

degeneration



our formula divisor $\cong \mathbb{F}_{a_i - b_i}$ fails exactly here: this divisor is

$$SL(2)/\mathbb{C}^* \cong \mathbb{P}^1 \times \mathbb{P}^1$$