

Linearizations, Realizations, Use, Detections

Historical analogy : the study of Fano varieties now is in much the same place as algebraic number theory was prior to the proof of the Weil conjectures in the 1970s.

Review :

Manin : ~~###~~ Correspondences, Motives, and... 1968

Deligne : The Weil Conjectures for K3 surfaces

The idea : 'linearize' algebraic geometry

Var $\xrightarrow[\text{correspondence}]{\text{Serre}}$ Linear Category

example (Mann)

$V =$ cubic 3-fold in $\mathbb{P}^4$

if we can make sense of

$$V = \mathbb{P}^3 + (\tilde{C} - C)(\mathbb{P}^1 - pt)$$

curves

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then the Weil conjectures would follow from the Weil conjectures for curves.

This was suggested by Grothendieck's idea of linearization : alg. geom should be turned into a branch of linear algebra.

Ideally: $\exists$ category of representations of a "group" G with

(1) operations $\oplus, \otimes$

(2) it should make sense to speak of a "cycle" $C \subset X$ (X = Representation here)
 $\uparrow$ subrep?

(3) algebraic over the base field $\iff G$ -invariant

Ideally : could be looking at a semisimple Tannakian category

Tannakian category = linear category where as



many properties are axiomatized as to force (3)
that C is equivalent to Rep_G for some
group ~~like~~ ^{like} object G (po-group, ...)

(unit object, duality operation,
  many more)

G can be recovered by ~~the~~ taking the
group of automorphisms of the fiber

functor (= functor to linear category
with appropriate properties).

Given $V_1, V_2 \ni$ projector at the linear level $V_1 \rightarrow V_2$,

What is the quantum version of all of this?

Consider $\text{Fano} \subset \text{Var}$

but with fewer morphisms.



We still dream of G^{quantum}

(4)

r.t. Fano $\xrightarrow[\text{not to}]{\text{gives}}$ rep of G^{quantum}

but quantum invariants are just some
special classes
of G -invariant

ie. G^{quantum} -invariants

So : G^{quantum} is bigger

fewer classes

fewer correspondences

fewer projections

than in the classical motivic picture.

The classically-split objects become irreducible
as reps of G^{quantum}

example : $\mathbb{P}^1$ is quantum irreducible



We know almost nothing at this point about the possible Tannakian approach.

But the situation is much better with realizations: these are ~~constructed~~ ^{suggested} by the constructions in quantum cohomology.

Classical:

Tate realization

Given $V/\mathbb{Q}$, consider the l -adic cohomology $H^{\text{ét}}(\bar{V}, \bar{\mathbb{Q}}_l)$ as a $\text{Gal}-\mathbb{Q}_l$ -module. The Tate

we are
in our
ideal world
here

→

Congruence holds, i.e. the Tate realization is a faithful functor

(i.e. if the Galois representations are the same then $\exists$ algebraic morphism (correspondence) between the two objects). Also the

Frobenius-Mazur Congruence holds:

suppose ρ is a cts $\text{Gal}_{\mathbb{Q}}$ -rep. s.t.

- (1) almost-everywhere unramified
- (2) is potentially s/s everywhere

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(3) B s/s

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then: ρ comes from geometry (it is the
Tate realization of a pure motive).

Thus can detect ab. varieties by detecting
their motives over a base field

↓
detects their Tate realizations as Galois reps

in practice: detect an L-function by

Selberg-Stark

(~~computationally feasible~~^{in principle} algorithmic)

e.g. even if I don't understand ab. geom. very well,
can find the elliptic curve $X_0(11)$ [of smallest
possible conductor] by systematically, algorithmically
looking for L-functions.

"Quantum Tate realization" is a rep of $\text{Gal } \mathbb{Q}(\mu_q)$
the $\otimes$ product is NOT $\otimes$ of reps but
rather convolving them w.r.t. $X^1 = \text{Spec } \mathbb{Q}[\mu_q]$.

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Th^m: (Katz) Perverse sheaves on A' with $\textcircled{7}$
 [mild technical property]



$k = \text{alg. closed of char} = p$

~~FT(k)~~
 FT(k) [-17]₇

is an equivalence of categories

middle-dim convolution $\longleftrightarrow \textcircled{x}$

$(X \mapsto -X)^*$

$\longleftrightarrow$ contragredient rep.

Quantum deletion:

In our search for Fano's * in odd dimension
 start with a search of self-dual
 self-dual Galois rep's of $\mathbb{Q}(q)$
 "of Euler characteristic zero"

gives Galois
 rep of $\mathbb{Q}(q)$

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take a 'Model' of the rep
then $\ast$ -extend.

⑧

(see next talk [=my talk] for practical
implementation in dimension 3)

One more analogy: Deligne proved
the Weil conjectures for K surfaces
by establishing the 'Satake construction'.

Th^m (Deligne): Let Y_0 be a K ,
polarized. Then $\exists$

(a) S of finite type $/\mathbb{R}$
 $Y \rightarrow S$ polarized K 's

(b) K' ext. of K and a point $\gamma: \text{Spec } K' \rightarrow S$
s.t. $\gamma^* Y / \simeq Y_0 \otimes_K K'$

(c) ! alg scheme $a: A \rightarrow S$



(d) $\mathbb{Z}$ -algebra $\mathbb{C}$ and $\mu: \mathbb{C} \rightarrow \text{End}_S(A)$

(e) no. of sheaves of $\mathbb{Z}_c$ -algebras over S

$$C^*(P^1 f_* \mathbb{Z}_c(i), \psi_c) \xrightarrow{\sim} \text{End}_C(P^1 \otimes)$$

Does there exist a parallel to this in the quantum situation?

Manivel:

classical

$[K]$

Quantum

even-dim quadric

