Introduction to the Satake Correspondence

joint work in progress with MANIVEL

GOLYSHEV

Start with two classical examples

1) behaviour of quantum motives under blow-up (Katzarkov, Smirnov)

2) Weil conjectures as proved by Deligne

The "spin rep. of K3" at the level of the Betti-deRham realization can be formalized as a rep ρ_{std} of $\mathrm{Spin}_{\mathbb{Q}}$, $\rho'_{\mathbb{R}} \to \mathrm{Spin}_{\mathbb{Q}} \otimes \mathbb{R}$

ρ_{spin} comes from geometry (Riemann)

Pattern: (common to classical and quantum)

alg. geom $\xrightarrow[\text{construction}]{\text{linear}}$ (1) does the result come from geometry?

(2) what can we prove on one side, knowing things on the other

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics

One possible incarnation of this approach:

Classical	quantum
K3 $\xrightarrow{\text{spin}}$ abelian	$\mathfrak{g}$ a simple Lie algebra
	$a \in \mathfrak{g}$
	ρ_1, ρ_2 finite-dim irreps
	basic setup throughout: $\mathbb{C}_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$, $D = t\partial_t$
	Take a connection
	(1) $Dh = \rho_1(a)h$
	suppose that (1) is quantum Fano geometric (i.e. arises from q.c. of a Fano)
	Now suppose further that
	(2) $Dh' = \rho_2(a)h'$

Q1 is (2) quantum geometric i.e. is it the Betti/de Rham realization of a piece of a Fano?

Q2 ~~[illegible]~~ can we prove things about Fano 2 given info. about Fano 1?

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics

<u>example</u> Let $\mathfrak{g} = so(n)$

a be characterized by $\rho_{std}(a) = (-K\star)$ in

q.c. of the quadric $OG(1, 2n)$. Put

$\rho_2 = \frac{1}{2}$-spin rep of so_{2n}.

$Dh' = \rho_2(a) h'$ <u>is</u> quantum and is the

quantum connection given by $-K$ in $OG(n, 2n)$.

Why start with $so(2n)$ and not $\mathfrak{sl}(2n)$

<u>Th$^{\underline{m}}$</u> (Bertram, Ciocan-Fontanine, Kim)

A similar picture holds for $G(i, n+1)$

with
$$\rho_1 = \rho_{std}$$
$$\rho_i = \cancel{}(\rho_{std})^{\wedge i}$$

Commonality: the Grassmannians in question are

<u>minuscule</u> (or maybe cominuscule?)

Type A situation: answer to Q2 is <u>yes</u>.

probably also in type D.

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics

The approach to the Tannakian properties above is to show that the Drinfeld-Ginzburg picture, or sub-picture of geometric Satake admits a 'quantum correction'.

What is geometric Satake?

G = semisimple over $\mathbb{C}$

$\mathbb{K} = \mathbb{C}((z))$ formal Laurent field

$\mathcal{O} = \mathbb{C}[[z]]$ lattice in it

$G(\mathbb{K})$ is a subgrp of $\mathbb{K}$-valued matrices

The "global version" of this is

$$LG = G(\mathbb{C}[z, z^{-1}])$$

$$L^+G = G(\mathbb{C}[z])$$

These groups are filtered:

$$L^+G = L^0 \supset L^1 \supset L^2 \supset \cdots$$

by considering loops f that are regular, $f(1) = 1$, and derivatives vanish up to order i.

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics

Seek
$$Gr = G(\mathbb{K})/G(\mathcal{O})$$
which is of infinite type but ~~[illegible]~~ can be approximated inductively

Prop^n: 1) Gr is the union of an infinite sequence of subsets Gr_i such that $Gr_i \subset Gr_j$ for $i<j$ and that are $G(\mathcal{O})$-stable and that.

2) $Gr_i \hookrightarrow Gr_j$ is a projective embedding

3) for each Gr_i the $G(\mathcal{O})$-action factors through some $G(\mathcal{O})/L^n$ for $n \gg 0$

Fact: $G(\mathbb{K})/G(\mathcal{O}) \cong LG/L^+G$

The dirty thing: convolution

We will take convolution of constructible sheaves on Gr; it is not a priori clear that this should be possible. There is a refined argument establishing this, which is quite difficult to follow, and also the following dirty argument.

Use the model $Gr = LG/L^+G$.
endow it with an $\mathbb{R}$-topological group structure by using a sort of Iwasawa decomposition:

K = maximal compact subgrp of G

$$\Omega := \{ f : S^1 \to K \mid f(1) = 1)$$

$$= \text{polynomial maps } f: \mathbb{C}^\times \to G \text{ s.t. } f(\bar{z}) = \overline{f(z)}$$

Iwasawa: $LG = \Omega \cdot L^+G$

$$\Omega \cap L^+G = \{1\}$$

so $Gr = LG/L^+G$ has a topological group structure

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics

Now

$$m : Gr_i \times Gr_j \longrightarrow Gr_k \qquad \text{for } k \gg 0$$

and then $*$ is defined on $D^b_c(Gr)$ by

$$L * M = m_*(L \boxtimes M)$$

<u>Th^m</u> : $*$ is associative.

There is a <u>left</u> action of L^+G on Gr. Each L^+G-orbit is contained in some Gr_i, $i \gg 0$, and is a smooth, locally-closed algebraic subvariety (in fact it is a v.b. over a partial flag variety for G). Each Gr_i is stratified by these orbits.

We have too many constructible sheaves on Gr, so we will consider the orbit intersection

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics

sheaves. Given an orbit $\mathbb{O}$ denote by $IC(\mathbb{O})$ the intersection complex on $\overline{\mathbb{O}}$ (i.e. take constant sheaf on $\mathbb{O}$ and then take Goresky-Macpherson extension to $\overline{\mathbb{O}}$)

Denote by $P(Gr)$ the abelian full subcategory of $D^b(Gr)$ with objects that are perverse sheaves isomorphic to sums of copies of $IC(\mathbb{O})$ over varying orbits $\mathbb{O}$.

Consider now the hypercohomology functor on P, $L \mapsto \mathbb{H}^\bullet(L)$.

<u>Th^m</u> (Ginzburg)

1) L is stable under convolution:

$$L, M \in P \Rightarrow L * M \in P$$

2) $(P(Gr), *)$ is a semisimple rigid category

(this means "almost Tannakian": Tannakian without fiber functor)

3) $M \mapsto \mathbb{H}^\circ(M)$ is an exact fully faithful functor $P(Gr) \to \underline{Vect}$

(i.e. is a fiber functor)

By general Tannakian formalism, this proves that $(P(Gr), *)$ is isomorphic to $Rep(G^*)$, for some (pro) group G^*.

<u>Fact</u>: $G^* = G^\vee$, Langlands dual group

Description of orbits by dominant coweights of G

$T \subset G$ maximal torus

$\lambda : \mathbb{C}^\times \to T$ a coweight determines the coset $\lambda L^+(G) \subset LG$ (by interpreting λ as a Laurent poly). Let $\mathcal{O}_\lambda$ be the orbit of $\lambda L^+ G$ in Gr. Then:

<u>Fact</u>: any L^+G-orbit in Gr is $\mathcal{O}_\lambda$ for some coweight λ, and $\mathcal{O}_\lambda = \mathcal{O}_\mu$ if iff? λ and μ are in the same Weyl-group orbit

<u>Th$\underline{\text{m}}$</u> (Ginzburg): There is an equivalence between the $\otimes$ categories $(P(Gr), *)$ and $(Rep(G^\vee), \otimes)$ which sends

$$IC(\mathcal{O}_\lambda) \longmapsto W_\lambda$$

rep with highest weight vector λ

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics

Under this identification, the hypercohomology functor $\mathbb{H}$ goes to the obvious functor $\mathrm{Rep}(G^\vee) \to \mathrm{Vect}$.

This is the Geometric Satake correspondence.

Will apply this to cominuscule Grassmannians, ~~are~~ regarded as orbits in $\mathcal{G}r$.

Ginzburg's theorem endows $H^\bullet($our Grassmannian$)$ with an action of $G^\vee$ (b/c $IC(\mathcal{O})$ here is just the constant sheaf).

Max-Planck-Institut für Mathematik
Bonn
Max Planck Institute for Mathematics