# The Apéry Class and the Gamma Class

Basic approach to classifying Fano varieties is analogous to the situation regarding algebraic varieties in the early 1970s (Grothendieck, Deligne)

$$\text{Classify Fanos} \longleftarrow \text{classify Fano quantum motives}$$

$\uparrow$

consider realizations

and classify / detect Fano motives via their realizations

probably too ambitious at the moment

if and when detected, recover invariants (geometric + topological invariants of the Fano or its pieces) from realizations

Approaches:

1) LG models of a special type

construct realizations from explicit pencils

2) Zagier: "Apéry-like recurrences with integral solutions"

"crystalline realization"

Here the setup is as follows. Imagine that you want to discover del Pezzos / G-del Pezzos

equations D2

reflection grp acting on del Pezzos

Make computer search for D2's whose solutions have integral coefficients in their power series expansions (or, more broadly G-functions: solutions with denominators that grow very slowly)

(3) "Betti realization"

classify Markov-type diophantine equations and hope that $\mathbb{Z}$-solutions of these correspond to the numerology of exceptional collections on Fanos.

(Here, VG thinks, the prospects for success are bleak)

Tannakian direction

Envision quantum motives as representations of some big quantum motives Galois group.
This may not succeed — it never worked out in classical algebraic geometry — but this line of thinking may nonetheless be very illuminating.

In classical alg. geom, two setups,

big Tannakian category
generated by all
smooth proj. varieties

\+ correspondences
between them

small Tannakian
category
generated by
a smooth
proper variety V

There are two big sets of problems implied by
this picture.

1 given something (a motive?)
(a bunch of ~~[illegible]~~ realizations)
that are quantum geometric
in the sense that they come from some
Fano or Fano-like object, does a particular
linear construction preserve geometricity?

e.g. $Sym^2$ or whatever

2: if the answer is yes, how are the Fanos $F_1$ and $F_2$ related?

[roughly speaking: "what is the $\otimes$ product of two Fano varieties" "what is $\mathrm{Sym}^2$ of a Fano" etc.]

Where to start?

The quantum intersection motive of Schubert cells of the affine Grassmannian (of GL or SL)

Problem: is it true that $QM(\mathcal{O}_\lambda)$ is the $\lambda$-rep of $QM(\mathcal{O}_{std})$

Recall that the affine Grassmannian for a simple or semisimple Lie group is built as follows:

$$LG = G(\mathbb{C}[t,t^{-1}])$$

$$L^+G = G(\mathbb{C}[[t]])$$

~~algebraic~~ $Gr = LG/_{L^+G}$

This is of infinite type but can be approximated by an inductive system of finite-type schemes $\{Gr_i\}$

The $LG_+$-orbits in $Gr_\lambda$ (or $Gr_i$) are in one-to-one correspondence with dominant coweights of $T \subset G$ (max torus)

Let $\lambda$ be a dominant coweight. Take the constant sheaf on the orbit $O_\lambda$ and Goresky-MacPherson extend it to $\overline{O_\lambda}$

Call this $IC^\bullet(O_\lambda)$ "intersection complex"

Have a hypercohomology functor $\mathbb{H}$ to <u>Vect</u>

<u>Classical Drinfeld-Ginzburg:</u>

$IH^\bullet(O_\lambda) = V_\lambda$ | where we regard $\lambda$ as a dominant <u>weight</u> for the Langlands-dual group $G^\vee$

This is the Satake correspondence

## Quantum Satake (joint work in progress with L. Manivel)

$$QM(\overline{O_\lambda}) = \underbrace{QM(\overline{O_{std}})}^{\lambda}$$

just a projective plane

std rep.

raised to the power $\lambda$ in the sense of plethysm

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break for questions:

Classically

catgys generated by the extension perverse sheaves of $O_\lambda$

⊗ operation = convolution

[see Baum talk]

this is Tannakian with fiber functor $\mathbb{H}(-)$ to Vect

Obstacles:

There is as yet no definition of quantum cohomology of $\overline{\mathcal{O}_\lambda}$

Is the principle true?

If yes, so what?

Suppose that $\lambda$ is miniscule, so that $\overline{\mathcal{O}_\lambda}$ is smooth ~~[illegible]~~ (in fact, $\overline{\mathcal{O}_\lambda}$ is smooth $\Leftrightarrow$ $\lambda$ miniscule)

$$\therefore \quad IQH(\overline{\mathcal{O}_\lambda}) = QH(\overline{\mathcal{O}_\lambda})$$

If the principle is true then ~~then~~ we have

Type A: $QM(G(k,n)) = \Lambda^k G(1,n)$

Type D: $QM(OG(n,2n)) = QM(OG(1,n))^{\text{half-spin}}$

Bertram–Ciocan-Fontanine–Kim
Hori–Vafa

Golyshev–Manivel
at the level of "de Rham realization"
i.e. quantum cohomology D-module

It has always been Deligne's principle not to mention motives, but only compatible systems of realizations. Let us follow that practice here too. What would it give us?

Two steps in this direction:

* Ueda's proof of Dubrovin's Conjecture for $G(k,n)$

* Galkin-Golyshev-Iritani: Apery = Gamma

The fiber product of these two results is the "extended Dubrovin Conjecture" (see later)

Why the Apery class? (related to rationality questions for Fano 3-folds)

The irregular monodromy datum according to Dubrovin (see e.g. his ICM talk)

Non-independent data:

1) a linear space $V$ [NB bad notation: here $V$ is not a variety]

2) a symmetric non-degen. form $\langle\, ,\, \rangle : V \otimes V \to \mathbb{C}$

3) a skew-symmetric operator $\mu : V \to V$
$\langle \mu a, b\rangle = -\langle a, \mu b\rangle$

4) a distinguished vector $e_1 \in V$ s.t. $\mu e_1 = -\frac{1}{2} d e_1$

5) $R : V \to V$ s.t. $R$ is $\mu$-nilpotent

i.e. $R_i : V_\lambda \mapsto V_{\lambda + i}$

and $R = R_1 + R_2 + \cdots + R_i + \cdots$

$V_\lambda$ = $\lambda$-eigenspace of $V_{\mathbb{C}}$ w.r.t. $\mu$

6) Stokes matrix on $\mathbb{C}^n$ [<u>not</u> $V$]

an upper $\Delta^{lar}$ matrix with entries $\langle a, b\rangle_{S'}$

7) central connection matrix $C: \mathbb{C}^n \to V$
such that $\langle a, b\rangle_{S'} = \langle Ca, e^{\pi i \mu} e^{\pi i R} Cb\rangle$

[maybe $C$ goes the other way, $V \to \mathbb{C}^n$
whatever]

This is the irregular monodromy data for the first structure connection in Dubrovin's theory. The structure connections for Fano mflds are quite special; let us see what this all becomes in that case.

<u>example</u>: for $QH(\mathbb{P}^d)$

$$V = H^\bullet(\mathbb{P}^d) = \oplus\, \mathbb{C} e_\alpha$$

$$\langle\ ,\ \rangle = \text{intersection form (Poincaré duality } \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}\text{)}$$

$$\mu = \tfrac{1}{2} \deg_{\text{usual}} - \tfrac{d}{2}$$

$$e_i = 1 \in H^\bullet(\mathbb{P}^d)$$

$$R = c_1 \bullet \qquad R e_\alpha = (d+1) e_{\alpha+1}$$

$$S_{ij} = \begin{cases} \binom{d+1}{j-i} & i \le j \\ 0 & \text{else} \end{cases}$$

Dubrovin proves that $C = C' \cdot C''$

where $C' = (C')^{\alpha}_{\beta}$

$C'' = (C'')^{\beta}_{j}$

and:

ASSUME $d$ is ODD

$$(C'')^{\beta}_{j} = 2\pi i \frac{(j-1)^{\beta-1}}{(\beta-1)!}$$

$$(C')^{\alpha}_{\beta} = \begin{cases} \dfrac{1}{\pi(d+1)} A_{\alpha-\beta}(d) & \alpha \geq \beta \\ 0 & \alpha < \beta \end{cases}$$

where the $A_j$ are the coefficients of the Laurent expansion of powers of the $\Gamma$-function:

$$\Gamma^{d+1}(-x) = \frac{1}{x^{d+1}} + \frac{A_1}{x^d} + \frac{A_2}{x^{d-1}} + \cdots$$

Generalize to any Fano as follows:

vector space, intersection form, identity, etc all directly as above

A simplified version: Shift perspective so the central connection is the main object

$V$ = vector space with semiorthonormal basis

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

(form is neither symmetric nor antisymmetric)

$V_1$ "$\otimes$" $V_2$ via $V_1 \otimes V_2$ and

$$[v_1 \otimes v_2, v_3 \otimes v_4] = [v_1, v_3][v_2, v_4]$$

This is a down-to-earth description of the semisimple part of the convolution category

This is <u>equivalent</u> to Dubrovin's data ~~1-7~~ (1-7)

and the $v_i$ are the rows/cols of central connection matrix [which?].

What is the generalization of the C' C''?
Dubrovin: the/a basis of an exceptional collection $ch(E_i) \cdot \gamma(\mathbb{P}^d)$ where $\gamma$ is the $\Gamma$-class.

$$C_j''^{\Gamma} = ch(\mathcal{O}(j-1))$$

$$C_j'^{\Gamma} = \text{basically } \Gamma\left(1 - \frac{x}{2\pi i}\right)$$

$\Gamma\left(1 - \frac{x}{2\pi i}\right) = 1 + \ldots$ so can produce the Hirzebruch mult. char. class

$$\Gamma(E) = \prod_{\substack{\text{Chern} \\ \text{roots } \rho_j \text{ of } E}} \Gamma\left(1 - \frac{\rho_j}{2\pi i}\right)$$

<u>So</u> : start with $\mathbb{P}^d$

construct quantum diff. eq$^n$

$$Dh = -K \star h$$

sol$^n$s of QDE $L^{\mathcal{Q}} = 0$

$\updownarrow$

homology classes

Monodromy of $L^{\mathcal{Q}}$ + the identification of the fiber with $H^\circ(\mathbb{P}^n)$ produces the central connection

$\longrightarrow$ SEE LATER? not explained now

Fano $\longleftrightarrow$ CY $\subset$ Fano anticanonical

irregular $\longleftrightarrow$ regular

thus stars (omitted here) $\longleftrightarrow$ 

pick a basis of vanishing cycles ...

## Extended Dubrovin Conjecture:

there exists a central connection matrix ~~[illegible]~~ with rows/cols equal

to $ch(E_i) \cdot \Gamma(V)$

for some exceptional collection

WHICH?

$\{E_i\}$ in $\mathrm{Ob}\, D^b_{coh}(V)$.

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In particular

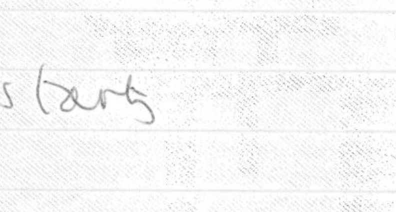
for all Fanos with exceptional collections that we know

(i) $\exists$ exceptional collection that starts

$$\{\mathcal{O}, \dots\}$$

(ii) it makes sense to say which row/col of connection matrix has the largest absolute value

cf regular situation



vanishing cycles are "marked in the base"

so it makes sense to speak of their absolute value

The largest one corresponds to $\mathcal{O}$ and is therefore the $\Gamma$-class.

So instead of considering the whole exceptional collection one can consider just a piece: the part $(\{\mathcal{O}\})$. $\rightsquigarrow$ largest crit-value (in absolute value) in LG potential

If that holds (ie the // transport of the vanishing cycle closest to $\infty$ is $\gamma(v)$) we say that Apery = Gamma.

## $A = \Gamma$ Hypothesis

**Th$^{m}$**: $A = \Gamma$ holds for $G(k,n)$.

also the extended Dubrovin Conjecture holds for $G(k,n)$.