KATZARKOV

It would not be a proper conference without at least one incomprehensible talk. So here goes

1) HMS
2) invariants
3) applications

(today: 19th applications
London: 20th applications)

$X_{\mathbb{C}}$ 3-dim'l cubic

field of algebraic functions

$\mathbb{C}(X) \not\cong \mathbb{C}(x_1, x_2, x_3)$

X smooth; Clemens-Griffiths

They approached this by studying the intermediate Jacobian

$$J(X) = H^{2,1}(X)^* / H_3(X;\mathbb{Z}) \supset \Theta \cup \Theta_{sing}$$

theta divisor

codim = 5

so $J(X)$ cannot be the Jacobian of a curve

$$\xRightarrow{\text{Griffiths's standard theory}} \quad \mathbb{C}(X) \not\cong \mathbb{C}(x_1, x_2, x_3)$$

Now will suggest analogs of this.

Bardelli: degenerated $X \rightsquigarrow X_0$

union of planes

$\Rightarrow \quad 1 \to (\mathbb{C}^\times)^2 \to J(X_0) \to \bigoplus_{i=1}^{3} E_i \to 1$

and the extension class here does not arise as the extension coming from a curve

$\Rightarrow \quad \mathbb{C}(X) \not\cong \mathbb{C}(X_1, X_2, X_3)$

X – smooth projective

$$D^b(X) \longleftrightarrow LG(X)$$

$LG(X)$ = moduli space of certain Lagrangians

$Y = LG(X)$

Fukaya-Seidel category

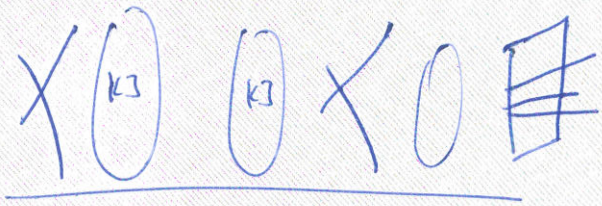
$FS(Y, f) = \begin{cases} \text{thimbles } L_i \\ \text{morphisms } HF(L_i, L_j) \end{cases}$

X = smooth cubic

$$f = \frac{(x_1 + x_2 + 1)^3}{x_1 x_2 y_1} + y_1$$

$(x_1, x_2, y_1) \in (\mathbb{C}^\times)^3$

Compactify: get family of K3 surfaces



2 ODP fibers; one v. complicated fiber

Now look at:

$$\begin{array}{c} LG(X) \\ \cup \\ LG(X)_{sing} \end{array}$$

have:

def. retract $\quad r_* : Y_t \to Y_0 \quad$ near singular fibers

and hence $\quad r_* : \mathbb{Z}_{Y_t} \to \mathbb{Z}_{Y_0} \in D^b_{const}(Y_0)$

which gives

$$\begin{tikzcd} r_* \mathbb{Z}_{Y_t} \arrow[rr] & & \mathbb{Z}_{Y_0} \arrow[dl] \\ & \mathcal{F} \arrow[ul] & \end{tikzcd}$$

exact $\triangle$

where $\mathcal{F}$ = perverse sheaf of vanishing cycles

$\dim \mathbb{H}^1 \mathcal{F} = 5$

$\dim \mathbb{H}^2 \mathcal{F} = 4$

$\dim \mathbb{H}^3 \mathcal{F} = 5$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & 1 & & & \\ 0 & & 5 & & 5 & & 0 \\ & & & 1 & & & \\ & & & 1 & & & \end{array}$$

Hodge $\diamond$ of cubi

This is part of a general theorem:

Theorem (Gross, Katzarkov)

X = Fano of dim ≤ 4, (flat def. to) toric c.i.

$\updownarrow$

can construct $LG(X)$, and $\mathbb{H}^\circ(\mathcal{F})[-1] \underset{\text{Hodge str.}}{\cong} H^\circ(X)$

rotate 90°

X = 3d Fano
rk Pic = 1
not of semisimple type

$\xleftrightarrow{\text{Przyzalkowski}}$

know LG mirrors
so take Y_0.
$H^2(Y_t)_{tr}$ transcendental part

X is rational $\xleftrightarrow[\substack{\text{V.G.} \\ \text{Golyshev}}]{\substack{\text{Przyzalkowski} \\ \text{V.P.}}}$ Monodromy action is purely unipotent

Empirical only: use Mori-Mukai list
Przyzalkowski calculation

Introduce now the invariant which explains this connection:

Spectra of triangulated categories:

$\mathcal{T}$ = triangulated category

Say $\mathcal{J}$ is a strong generator for $\mathcal{T}$ if

$$\langle \mathcal{J} \rangle_{\substack{\text{sums}\\ \text{summands}\\ \text{shifts}\\ \text{triangles}}} = \mathcal{T}$$

$t(\mathcal{F})$ = generation time

= maximal # of $\triangle$s needed to get from $\mathcal{F}$ (and summands and shifts) to any other element of $\mathcal{T}$.

For (G_i) strong generators, have

$$\operatorname{Spec}\mathcal{T} = (t(G_1), t(G_2), \dots) \qquad \text{(Orlov)}$$

Min # =: $\dim(\mathcal{T})$
(Rouquier)

Now how to compute and how to connect to ~~ncHS~~ non-commutative Hodge structure (ncHS)

example: $D^b(E) = \mathcal{T}$

$\mathrm{Spec}(\mathcal{T}) = (1, 2, 3, 4)$

Thm (Ballard-Favello-Katzarkov)

$$
\left.\begin{array}{l}
1 \longleftrightarrow \mathcal{O}_{-3p} + \mathcal{O} + \mathcal{O}_{3p} \\
2 \longleftrightarrow \mathcal{O} + \mathcal{O}_{3p} \\
3 \longleftrightarrow \mathcal{O} + \mathcal{O}_{2p} \\
4 \longleftrightarrow \mathcal{O} + \mathcal{O}_{p}
\end{array}\right\}
$$

there are other possibilities too

By Mirror symmetry, this equals $\mathrm{Spec}(\mathrm{Fuk}(E))$.

example

$D^b(\mathbb{P}^2)$ $\qquad \mathrm{Spec}(D^b(\mathbb{P}^2)) = (2, 3, 4)$

$2 \hookrightarrow \mathcal{O} + \mathcal{O}(1) + \mathcal{O}(2)$

$3 \hookrightarrow \mathcal{O} + \mathcal{O}(1) + \mathcal{O}_C$

$4 \hookrightarrow \mathcal{O} + \mathcal{O}_C + \mathcal{O}_p$

<u>example</u> $\mathcal{T} = D^b_{sing}(\mathbb{C}, x^n)$

$$\mathrm{Spec}(\mathcal{T}) = (0, 1, \ldots, n-1)$$

$$\mathrm{Spec}(\mathcal{T}/\mathbb{Z}_8) = (0, 1, 3) \qquad \text{GAP!}$$

for $n=8$

<u>Theorem</u>: X smooth projective

$\mathcal{T} = D^b(X)$

$\mathrm{Spec}\,\mathcal{T}$ is bdd

<u>Thm</u>: X is rational of dim n $\Rightarrow$ $\mathrm{Gaps}(\mathrm{Spec}(\mathcal{T})) \leq n-2$

[Convention: if no missing #, gap $= 0$]

FROM NOW ON: Conjectures only.

Return to our example. Using LG model, compute (conjectural) spectrum of cubic

$$\mathrm{Spec}(D^b(X)) = \langle 3, 4, \dots, 13, 14, 16 \rangle$$

$X = 3d$ cubic

GAP

Geography of 4-dim'l cubics :

X = 4d cubic	Gap $\mathrm{Spec}(D^b(X))$	Automorphic forms
generic	3	
X contains a plane	2	
other Noether-Lefschetz loci	Gap drops too	$\{$Heegner divisors$\} \longleftrightarrow \{$Noether-Lefschetz loci$\}$ $\{$drop in gaps$\}$