Quantum Cohomology of Miniscule type D Grassmannians

Consider OG

<u>notation</u>: $\mathbb{C}^{2n}$ has standard symmetric i.p.

$$\langle e_i, e_j \rangle = \delta_{i+j,\,2n+1}$$

$\Sigma \subset \mathbb{C}^{2n}$ isotropic $\Rightarrow$ $\Sigma \subset \Sigma^{\perp}$

dim $= m$ $\Rightarrow$ dim $= 2n-m$

so max isotropic subspaces are of dimension $m = n$.

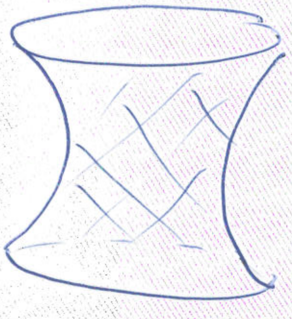
Geometric interpretation

$q = 0$, q = quadratic form associated to $\langle\, , \rangle$

defines a quadric $Q \subset \mathbb{P}^{2n-1}$

isotropic m-planes $\longleftrightarrow$ $\mathbb{P}^{m-1} \subset Q$

maximal isotropic $\longleftrightarrow$ $\mathbb{P}^{n-1} \subset Q$



two families of such

Def$^{\underline{n}}$ The <u>Orthogonal grassmannian</u> $OG(m,2n)$ is the space of isotropic m-planes in $\mathbb{C}^{2n}$, except that when $m=n$ (the maximal isotropic case) we take just one of the two families; by convention we take the one containing $F_n \subset \mathbb{C}^{2n}$

F_n: fixed favourite isotropic n-plane

<u>Equivalently</u>:

$$OG(n,2n) = \left\{ \Sigma \text{ max isotropic} \;\middle|\; \begin{array}{l} \Sigma \cap F_n \text{ has even} \\ \text{codim in } F_n \end{array} \right\}$$

<u>Notation</u>. Henceforth $OG = OG(n,2n)$ as we concentrate on this case.

Fix, for convenience, an isotropic flag $F_1 \subset F_2 \subset \cdots \subset F_n$

<u>Schubert varieties</u> in OG are indexed by strict partitions (ie no repeated parts) The partition

λ must have $\lambda_1 \leq n-1$. Then

$$X_\lambda = \{ \Sigma \in OG \mid \dim(\Sigma \cap F_{n-\lambda_i}) \geq i \}$$
for $1 \leq i \leq \cancel{} \ell(\lambda)$

<u>Schubert class</u> $\tau_\lambda = [X_\lambda] \in H^*(OG; \mathbb{Z})$

particular case $\tau_i = [X_i] = [\{ \Sigma \in OG \mid \Sigma \cap F_{n-i} \neq \{0\} \}]$

These are <u>special Schubert classes</u>. they generate the cohomology ring and there are good combinatorial formulas for how they act on the cohomology ring. (This is all parallel to the story for usual Grassmannians)

<u>Geometric interlude</u> this will also explain some indexing conventions

There is the universal bundle sequence:

$$0 \to S \to \mathcal{O}_{OG}^{\oplus 2n} \longrightarrow Q \to 0$$

and the pairing $\langle , \rangle$ descends to give a perfect pairing $S \otimes Q \to \mathcal{O}$.

Thus $Q \xrightarrow{\sim} S^\vee$ and

$$c_i(S^\vee) = \pi_*\left(c_1\left(\mathcal{O}_{\mathbb{P}(S)}(1)\right)^{i+n-1} \right)$$

Here

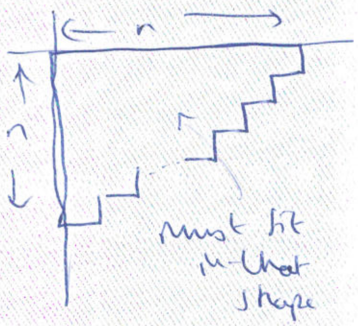
$$\begin{tikzcd}
\mathbb{P}(S) \arrow[rr] \arrow[d, "\pi"'] \arrow[dr, dashed] & & \mathbb{P}^{2n-1} \\
OG & Q \arrow[ur, dashed] &
\end{tikzcd}$$

quadric from before

Indeed:

$$c_i(S^\vee) = \begin{cases} 2\tau_i & 1 \le i \le n-1 \\ 0 & i = n \end{cases}$$

So now, to avoid too many $(n-1)$s, henceforth $OG = OG(n+1, 2n+2)$. Then we have special Schubert classes $\tau_1, \dots, \tau_n$, and general Schubert classes τ_λ with λ a strict partition with $\lambda_1 \le n$.



$$T_{OG} = \Lambda^2(S^\vee)$$

$$c_1(T_{OG}) = n\, c_1(S^\vee) = 2n\tau_1$$

So:

$$\dim OG = \frac{n(n+1)}{2}$$

$$\deg q = 2n \quad \text{(quantum parameter)}$$

$$\dim H^*(OG) = 2^n$$

where $\deg H = 1$

H = hyperplane

Ring presentation:

$$H^*(OG, \mathbb{Z}) = \mathbb{Z}[\tau_1, \dots, \tau_n] \Big/ (\quad)$$

$$(\tau_1^2 - \tau_2,\ \tau_2^2 - 2\tau_1\tau_3 + \tau_4,\ \dots,\ \tau_{n-1}^2 - 2\tau_{n-2}\tau_n,\ \tau_n^2)$$

For degree reasons, the only place that the quantum parameter can only occur in the degree n relation:

$$QH^*(OG, \mathbb{Z}) = \mathbb{Z}[\tau_1, \ldots, \tau_n] \Big/ (\quad)$$

$$(\tau_1^2 - \tau_2,\ \tau_2^2 - 2\tau_1\tau_3 + \tau_4,\ \ldots,\ \tau_{n-1}^2 - 2\tau_{n-2}\tau_n,\ \tau_n^2 - q)$$

The connection to algebraic combinatorics

There is a family of symmetric polynomials called $\tilde{P}$-polynomials, indexed by partitions, such that (Pragacz-Ratajski '97) if $x_1, \ldots, x_{n+1}$ are the Chern roots of the universal quotient bundle then

$$\tau_\lambda = \tilde{P}_\lambda(x_1, \ldots, x_{n+1})$$

for strict partitions λ, and

$$\tilde{P}_\lambda(x_1, \ldots, x_{n+1}) = 0$$

whenever λ is a non-strict partition

Furthermore

$$\tilde{P}_i = \frac{1}{2} e_i(x_1, \ldots, x_{n+1})$$ elementary symmetric functions

$$\tilde{P}_{ij} = \tilde{P}_i \tilde{P}_j + 2 \sum_{k=1}^{j-1} (-1)^k \tilde{P}_{i+k} \tilde{P}_{j-k} + (-1)^j \tilde{P}_{i+j}$$

$$\tilde{P}_\lambda = \text{Pfaffian}\left(\tilde{P}_{\lambda_i \lambda_j} \right)_{1 \le i,j \le 2\left\lfloor \frac{\ell(\lambda)+1}{2} \right\rfloor}$$

extend to even-size matrix by including a "fake" extra zero piece of λ if necessary

Giambelli formula follows by evaluating on Chern roots.

Now what happens if you have repeated parts? Taking the partition with ~~[illegible]~~ a single repeated part:

$$\widetilde{P}_{ii} = \frac{1}{4} e_i(x_1^2, x_2^2, \ldots, x_{n+1}^2)$$

elementary symmetric function of the squares

The relations in $H^*(OG)$ say exactly $\widetilde{P}_{ii} = 0$ (and in fact this also implies semisimplicity of QC: see Chaput-Manivel-Perrin).

Quantum Schubert calculus is usually understood as Giambelli + Pieri, so we now discuss Pieri.

Quantum Schubert calculus (Kresch-Tamvakis '04)

quantum Giambelli $\rightsquigarrow$ same as above

quantum Pieri

$$\tau_i \tau_\lambda = \sum_{\substack{\mu > \lambda \\ |\mu| = |\lambda| + i}} 2^{\#\mathrm{conn}(\mu/\lambda) - 1} \tau_\mu + q$$

$\to$ μ/λ is a horizontal strip

so add i boxes at most 1 box per column

quantum correction; see overleaf

$$\boxed{//\ell_i} = \sum_{\substack{\mu > \lambda,\ \mu \supset (n,n) \\ |\mu| = |\lambda| + i \\ \mu/\lambda \text{ a horizontal strip}}} 2^{\#\mathrm{conn}(\mu/\lambda) - 1} \tau_{\mu - (n,n)}$$

Two generalizations: $\rightarrow$ different Lie types

$\searrow$ non-maximal $OG(n, 2n+2)$

many things fail, but the quantum ring presentation still has

$$QH^* = \mathbb{Z}[\tau_1, \ldots, \tau_n] \Big/ \big(\qquad \big)$$

classical relations but with q in exactly one place