MANIVEL

1

Quantum Satake for Miniscule Spaces

Recall from Golyshev's talk:

affine Grassmannian Gr for G a simple Lie group

Ginzburg:
$$IH(\mathcal{O}_\lambda) \cong V_\lambda^{G^\vee}$$
λ a dominant coweight of G

Something that acts naturally on both sides: $H^*(Gr)$

By Bott, $H^*(Gr) \simeq \mathcal{U}(\mathfrak{u})$ for $\mathfrak{u}$ an abelian subalgebra of $\mathfrak{g} = Lie(G)$, $\mathfrak{u} = C(X)$, X = regular nilpotent

Geometric Satake intertwines the $H^*(Gr)$-actions.

In general the affine Schubert varieties $\mathcal{O}_\lambda$ are very singular, so one needs to consider the intersection cohomology, but the smooth $\mathcal{O}_\lambda$ are precisely the miniscule homogeneous spaces G/P.

$$H^*(G/P) \simeq V \quad \text{G-module}$$
$$\circlearrowleft \; \mathcal{U}(\mathfrak{u})$$
$X \in$

acts as hyperplane class here

$G = G^\vee$ in the miniscule case

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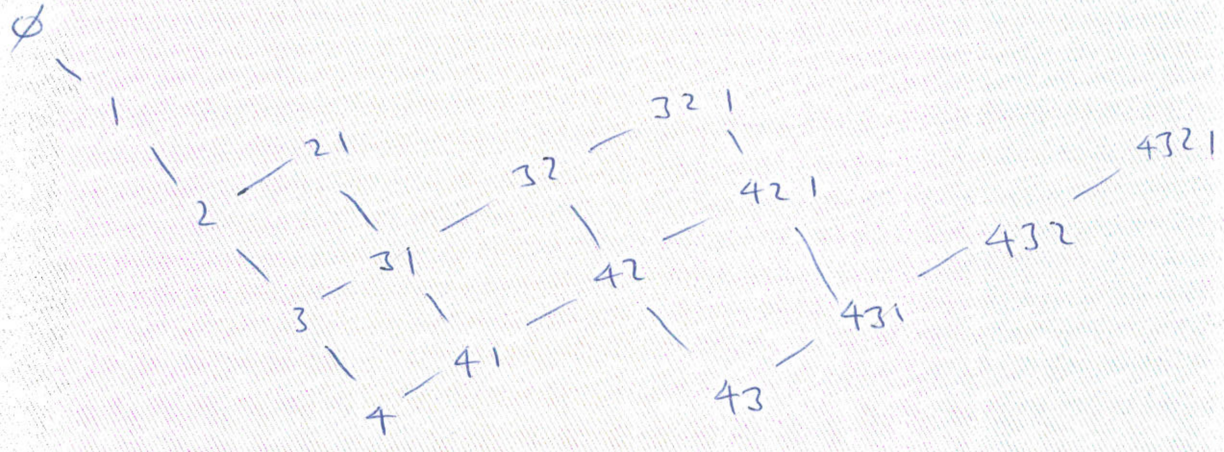
example $OG(5,10)$

Schubert classes τ_λ, λ a strict partition parts ≤ 4

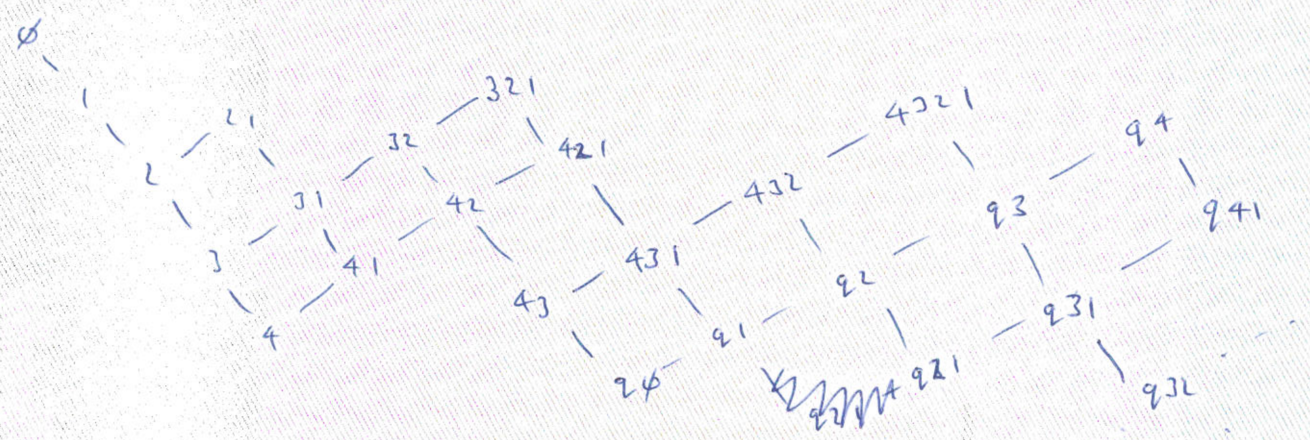


Hasse diagram

$$\tau_\lambda * H = \sum_{\lambda \to \mu} \tau_\mu$$

or, equivalently, this is the incidence relation between Schubert variety

Have quantum corrections:



now it is periodic

Satake : classical mult. by H

$$\text{action of } X = \sum_{\alpha \text{ simple}} X_{-\alpha}$$

(preferred) regular nilpotent

$\mathfrak{h}$ Cartan

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Phi \\ \text{roots}}} \mathfrak{g}_\alpha \ni X_\alpha$$

Quantum multiplication

Quantum Chevalley formula (Fulton–Woodward)

$$\text{quantum mult. by } H \simeq \text{action of } X_q = \sum_{\alpha \text{ simple}} X_{-\alpha} + qX_\psi$$

where ψ = highest root

This is not explained by the classical Satake correspondence. But as we will see, the whole story generalizes to the quantum setting:

$$QH^*(G/P)_{q=1} \simeq V$$

acts as hyperplane class $\curvearrowright$ $\circlearrowleft$ $X \in \mathcal{U}(\mathfrak{h}_q)$ $\qquad \mathfrak{h}_q = C(X_q)$

To explain this, first recall what is known about these spaces

For any semisimple Lie group G, $P < G$ parabolic

$$H^*(G/P, \mathbb{Z}) = \bigoplus_{\lambda \in W/W_P} \mathbb{Z}\, \sigma_\lambda$$

σ_λ = Schubert classes

W = Weyl group of G

W_P = Weyl group of P

example $\quad G = SL(n)$

$$P = \begin{array}{c c} & k \\ k & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}$$

$W \simeq S_n$ symmetric group $\qquad W/W_P \simeq \left\{ \begin{array}{l} \text{partitions fitting} \\ \text{in a } k \times (n-k) \text{ box} \end{array} \right\}$

$W_P \simeq S_k \times S_{n-k}$

example $\quad$ ~~SO_{2k}~~ $G = SO_{2n}$ $\qquad P = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$

$W = S_n \ltimes \mathbb{Z}_2^{n-1}$ $\qquad W/W_P \simeq \mathbb{Z}_2^{n-1}$

$W_P = S_n$

Have also the <u>Borel presentation</u>

$$H^*(G/P, \mathbb{C}) = \mathbb{C}[\mathfrak{h}]^{W_P} / I \qquad \mathfrak{h} = \text{Cartan}$$

where I = ideal generated by W-symmetric polynomials with no constant term.

The connection between these two descriptions is not so convenient.

$$I = \langle I_{d_1}, \ldots, I_{d_n} \rangle$$ where I_{d_j} are homogeneous polynomials of degree $d_1, \ldots, d_n$

The degrees d_j are called the <u>exponents</u> of the group

In the miniscule setting there are various coincidences.

<u>Def^n</u>. A G-module V is miniscule if the weights of $\mathfrak{h}$ are in a single W-orbit.

<u>Def^n</u>: G/P is miniscule if the minimal G-equivariant embedding $G/P \hookrightarrow \mathbb{P}(V)$ has V miniscule

There are not so many miniscule homogeneous

Spaces

<u>List</u>: Type A $\quad G(k,n) \subset \mathbb{P}(\Lambda^k \mathbb{C}^n)$ (Plücker)

Type D $\quad Q^{2n}$, $OG(n+1, 2n+2)$

Type E $\quad$ 2 examples

In the miniscule case, $V = \bigoplus_{w \in W/W_P} V_{w\lambda}$ ← 1-dim'l

so there is a coincidence of dimensions

$$H^*(G/P) = \bigoplus_{w \in W/W_P} \mathbb{Z}\sigma_w \qquad V = \bigoplus_{w \in W/W_P} V_{w\lambda}$$

<u>Quantum cohomology</u>: A priori we know that $\exists$ presentation of QC with relations which deform the relations in classical cohomology.

$$QH^*(G/P) = \mathbb{C}[h]^{W_P}[q] \Big/ \langle I_{d_1}, \ldots, I_{d_{i-1}}, I_{d_i} - q \rangle$$

↑ only deform the last relation

Consequence: $QH^*(G/P)_{q=1}$ is semisimple

<u>Gorbunov–Petrov</u>: This work (= Chaput, Manivel, Perrin) has close connection to work of Kostant from 1959.

$$d_i + d_{n+1-i} = h+1 \quad \text{where } h = \text{Coxeter } \#$$
$1 \le i \le n$

Cyclic elements

$$\mathbb{C}[h]^W \simeq \mathbb{C}[g]^G$$
$$I_{d_i} \longrightarrow I_{d_i}$$

nilpotent elements $X \longleftrightarrow I_{d_i}(X) = 0 \ \forall i$

Kostant defined the notion of a cyclic element X: these have $I_{d_i}(X) = 0$ for all i except the last one.

He proved that any cyclic element is Weyl-conjugate to:

$$X_q = \sum_{\alpha \text{ simple}} X_{-\alpha} + q X_\chi$$

$$\Rightarrow \quad QH^*(G/P)_{q=1} = \mathbb{C}[\underline{W.X_q}]^{W_P}$$

Fact: X_q is regular and semisimple

$C(X_q)$ = Cartan subalg., so $W \circlearrowright C(X_q)$. Now $X_q \in C(X_q)$

$\Rightarrow$ has Weyl orbit $W.X_q$

So $QH^*(G/P)_{q=1}$ = algebra of functions on W_P-classes in $W.X_q$

Now we come to Satake

Classical Satake

$\exists\ u \overset{\Theta}{\hookrightarrow} H^*(G/P) \quad \sigma_w$

$$H^*(G/P) \cong V, \qquad \sigma_w \mapsto v_w \in V_{w\lambda}$$

Such that $\forall\, y \in u$, $\Theta(y) \cup \sigma_w = y(v_w)$

so cohomological action of classes from $u \longleftrightarrow$ Lie algebra action on V.

This also works at the quantum level.

<u>example</u>. $G(k,n)$

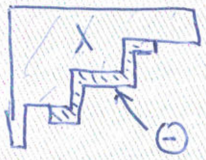
$G = SL_n$ $\qquad X = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$

$u = C(X) = \langle X, X^1, \ldots, X^{n-1} \rangle$

$\Theta(X^i) = p_i$ $\qquad$ the cohomology class defined by the i^{th} power sum

(recall $H^*(G(k,n))$ = quotient of the space of symmetric functions)

$$p_i \, s_\lambda = \sum_{\substack{|\theta| = i}} (-1)^{ht(\theta)} s_{\lambda + \theta}$$



s_λ: Schur function

θ = connected row hook

$ht(\theta)$ = # rows -1

This is classical. see Macdonald's book.

In the quantum setting, $\exists$

$$\Theta_q \quad h_q \longrightarrow QH^*(G/P)_{q=1}$$

$h_q = C(X_q)$, $\qquad QH^*(G/P)_{q=1} \cong V$

$$\Theta_q(Y) * \sigma_w = Y(v_w) \qquad [\text{as before}]$$

How to prove this?

$$QH^*(G/P)_{q=1} \simeq V$$

σ_w Schubert classes	e_w	weight basis for h
ι_ξ idempotents (b/c semisimple)	j_ξ	weight basis for h_q

Proof has two steps:

$$\sigma_w \xleftrightarrow{\varphi} e_w$$

$$\iota_\xi \xleftrightarrow{\psi} j_\xi$$

1) the identification through ψ gives

$$\Theta_q(Y) * \iota_\xi = Y(j_\xi) = \xi(Y)\, j_\xi$$

$Y \in$ Cartan $\Rightarrow$ just acts via weight

2) φ and ψ can be made equal

<u>example</u> for the spinorial representation

$OG(n, 2n)$

$$h_q = \langle X_q^{2k+1}, Y \rangle$$

$$\Vert$$

$$C(X_q) \qquad \downarrow \Theta$$

$$\langle p_{2k+1}, \tau_{n-1} \rangle$$

Then can explicitly match everything:

$$\rho_{2l+1} * \tau_\lambda = \sum_{\substack{\theta \text{ a row-hook} \\ \text{(connected)}}} (-1)^{ht(\theta)} \tau_{\lambda+\theta}$$

explicit sign

$$+ \sum_{\substack{\beta \text{ a double} \\ \text{row hook}}} (-1)^{\varepsilon(\beta)} \tau_{\lambda+\beta}$$

explicit sign

$$+ 2q \sum_{\substack{\gamma \text{ a double} \\ \text{row hook}}} (-1)^{\varepsilon(\gamma)} \tau_{\lambda-\gamma}$$

Classical formula is in Macdonald ; quantum correction is given by Satake