

A Tropical View on Landau-Ginzburg Models

joint work with Michael Carl, Max Pumperla

hopefully on the web very soon

application of work with Mark Gross

Overview: on the tropical approach to mirror symmetry

Don't start with fixed data (CY, Fano, whatever)
but instead start with a point of maximal degeneration
(for CY this is a point of maximal unipotent monodromy)

This makes sense physically: much of the mirror symmetry story is only defined near the large-volume limit.

What do we mean by "Fano"?

- K_X effective

$\check{X}$ = our variety

LG model $(X, f) \quad \updownarrow \quad f \in G_X$

Need maximal degeneration: (maximally unipotent)

this refers to degenerations of Hodge str. around the degeneration point

$(\check{X}, \check{\omega})$

$\downarrow$

$T = \text{Spec}(\text{dvr})$

or, more explicitly

Need also a polarization:

L relatively ample

$$(\check{X}, \check{D})$$



$$T = \text{spec } k[t]$$

discrete data $(\check{B}, \check{P}, \check{c})$ built as follows.

$\check{B}$ = topological manifold with affine structures
 $/ \mathbb{Z}$ away from $\Delta \subset \check{B}$

codim
= 2 $\mathbb{P}^2$

Δ = discriminant

$\check{P}$ = polyhedral decomposition $/ \mathbb{Z}$

transition f's
are ~~in~~
in $\mathbb{Z}^n \rtimes GL(n, \mathbb{Z})$

$\check{c}$ = strictly convex PL function
 (possibly multi-valued) on $(\check{B}, \check{P})$

How to get them: $(\check{X}, \check{D}, L) \rightsquigarrow (\check{B}, \check{P}, \check{c})$

$$\check{B} = \bigcup_{\sigma \in \check{P}} \sigma \quad \{\sigma\} = \{\text{momentum polytopes of } (X_\sigma, L)\}$$

example

$$V(4) \subset \mathbb{P}^3$$

$\rightsquigarrow$
degen.

$$X_0 = V(x_0 x_1 x_2 x_3)$$

e.g. $x_0 x_1 x_2 x_3 + t f_+(x_0, \dots, x_3)$

This gives affine structures
 on the interiors of σ ;
 what about near the



momentum
polytopes of
 $\mathbb{P}^2$ stuck together

(4)

vertices of $\check{P}$

this makes sense b/c
 $X_0 = \bigcup$ toric varieties
 glued torically

Assume Z misses the 0-dim' toric strata of X_0

π toroidal near $v \in \check{P}$ (vertex)

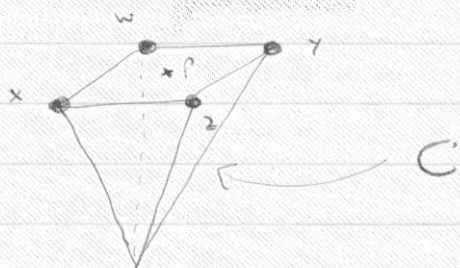
$$\pi^* : t \mapsto z^t \in k[Q]$$

$$p \in Q$$

$$Q = \bigcap \mathbb{Z}^{n+1}$$

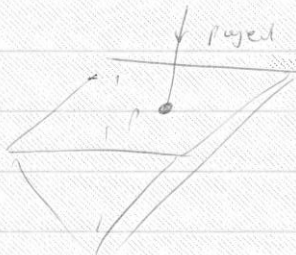
↑
rational polyhedral cone

example if local model is $xy - zw = 0$, $t \mapsto xy$

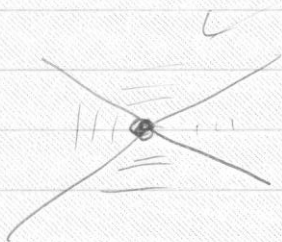


$$Q = \bigcap \mathbb{Z}^{n+1}$$

Use p -direction to project down:

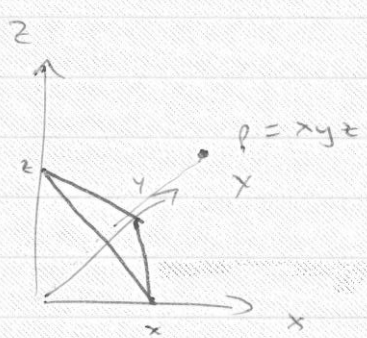


these "wedges" are part of the
polytope

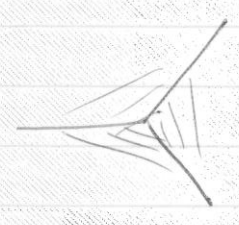


affine structure
is here

Example back to $V(4)$ example



projection :



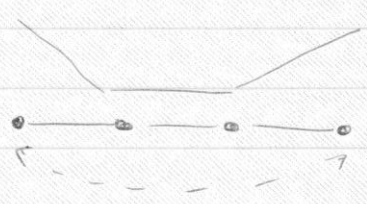
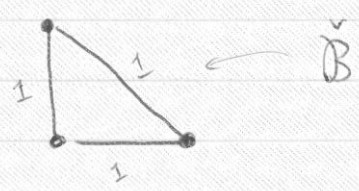
and define $\check{c}$ such that $\partial Q = \Gamma_{\check{c}}$ graph of $\check{c}$ at v
 or rather $\partial(Q \otimes \mathbb{R}) = \Gamma_{\check{c}}$

example : one dimension lower

$$\mathcal{L} = x_0 x_1 x_2 + t(x_0^3 + x_1^3 + x_2^3) = 0$$

$$L = \mathcal{O}_{\mathbb{P}^2}(1)$$

restricted to curve $\mathcal{L}$



∇ convex
 PL function
 b/c Δ closes up

} regard as equivalent



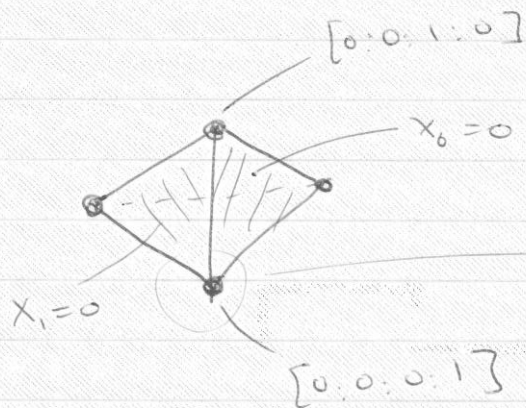
add global affine $f \geq$

just want "positive slope change" at each point

i.e. here $\mathbb{N}^3$
 (a, b, c)



back to $V(4)$ example

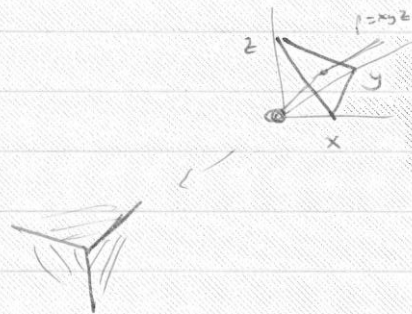


to get affine
st. near here
work in
affine chart

$$x_i(p_i) = 1$$

i.e. $[x, y, z, 1]$

and then do the



This is "the Eric data". You can go backwards from this data to a degeneration, too, as follows.

Reconstruction Theorem: (Cox-Siebert 2007)

Given $(\tilde{B}, \tilde{p}, \tilde{c})$ satisfying certain natural combinatorial conditions which I will not explain here:

- * possibility
- * simple local monodromy of the affine str.

monodromy in e.g. $n=2$
(monodromy about Δ)



$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = A$$

A is always of the form
 $A = Id + \langle u, - \rangle v$

Simplicity means each monodromy $w.$ is simple in the group theoretic sense and also some extra stuff; i.e. roughly speaking simplicity means "monodromy is indecomposable".

Then, given the conditions above, get (family of)

$$(\tilde{X}, \tilde{D}), L$$

↓

+

~~leading~~ which gives rise to $\tilde{D}, \tilde{P}, \tilde{c}$

oops! forgot to define
 $\Delta \subset \tilde{D}$

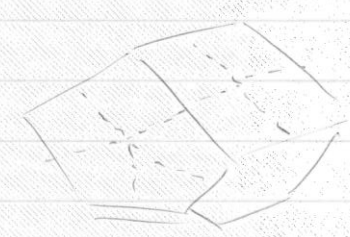
remove a canonical subset of $\tilde{D}$ s.t. rest is ~~is~~ covered by the affine charts from the vertices and affine charts from interiors of cells

defined in terms of barycentric subdivision

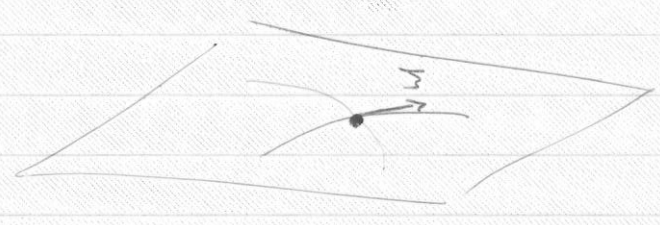


in $\dim=2$
just remove center of edges

in $\dim=3$



Uniqueness?

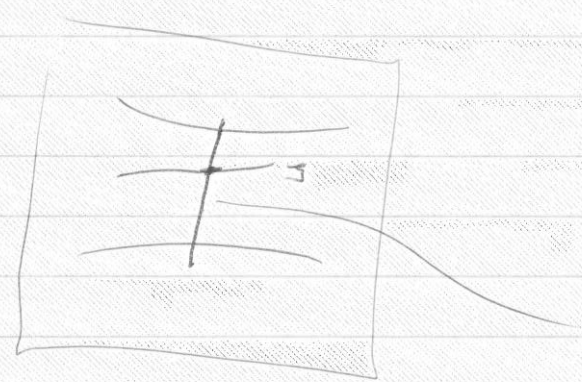


← ban of versal deformation of (X_0, D_0)

tangent vector S here is in $H^1(\tilde{B}, \omega \otimes_{\mathbb{Z}} \mathbb{C}^x)$

some finite-dim. grp computable in terms of affine geometry

This means



↑ this group classifies log structures on the central fiber

fiber is canonically $H^1(\tilde{B}, \dots)$ and away from the strict transform of the distinguished bad curves, given

↓ (weighted) blow-up



certain distinguished directions; not explained here

unique

12 laws

Siebert

$\exists!$ deformation over $T = \text{Spec } k[[t]]$ realizing S .

i.e. existence of log structure implies unobstructed Yes

Alessio : but that is very surprising, because there are many arcs that give the same tangent vector so this suggests that there is a ~~flat structure~~ flat structure on the base which distinguishes this unique arc.

Siebert : yes : we have somehow found the "canonical co-ordinates" ~~that~~ coming from mirror symmetry and this singles out the unique arcs : they are in canonical co-ordinates

Golyuker : Do you have a smoothness criterion in terms of $(\tilde{B}, \tilde{p}, \tilde{e})$?

Siebert : Yes. OK arises from $\mathbb{Z}$ (b/c toroidal) but also have local models near $\mathbb{Z}$.

~~Upshot: smooth iff moment map polyhedron are standard~~
(nb $\exists$ non-standard elementary polytopes)

only lattice points
are
vertices

e.g.
(0,0,0)
(1,0,0)
(0,1,0)
(1,1,a)

iff monodromy polytopes
are standard

not defined yet

elliptic curve

Recap: ~~old~~ example

at speed

$$\check{X} = X_0 X_1 X_2 + t(X_0^3 + X_1^3 + X_2^3) \subset \mathbb{P}^2 \times \mathbb{A}^1$$

$\check{D} = \emptyset$ (b/c this is honest CY)

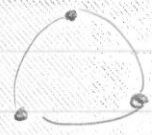
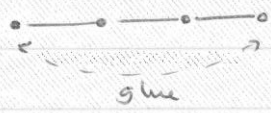
$$L = \mathcal{O}_{\mathbb{P}^2}(1) \big|_{\check{X}}$$

central fiber

$$\check{X}_0 = \mathbb{P}^1 \cup \mathbb{P}^1 \cup \mathbb{P}^1$$

all polarized by L
which restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$

$$\Rightarrow (\check{B}, \check{P}) =$$



after s'
of length 3

This is a normal crossings ~~interaction~~ ^{degeneration}

$\Rightarrow \check{C}$ changes slope by 1 at any v

b/c smoothings of each
node $\hookrightarrow xy = t$
not $xy = t^m$ or whatever
 $\Rightarrow$ slope change 1 not
slope change m

Now run the program to get

$$\begin{array}{c} \check{X}' \\ \pi' \downarrow \\ T = \text{Spec } k[[q]] \end{array}$$

coming actually from

$$\begin{array}{c} \check{X}'' \\ f \downarrow \text{holomorphic} \\ \mathbb{D} \end{array}$$

analytic
unit disc

$$f^{-1}(q) \simeq \mathbb{C}^x / \langle q^{1/3} \rangle$$

nb we ~~have to do this~~
~~normally b/c~~
~~think about~~ $q \rightarrow$

Landau - Ginzburg Models

Suppose now that $\check{Z} \neq \emptyset$

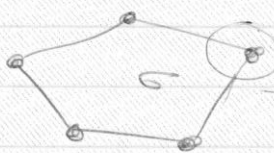
This implies $\partial \check{B} \neq \emptyset$

Remark: $(\check{X}, \check{Z})$ inherits the singularities along $\check{Z}$ from $\check{B}$ at $\partial \check{B}$

example $V(2) \subset \mathbb{P}^1$
 $\rightsquigarrow$ degenerates to $\mathbb{P}^1 \cup \mathbb{P}^1 \dashrightarrow$
 doesn't close up

example:

$$\check{B} =$$



$$\check{P} = \{ \sigma, \text{ faces of } \sigma \}$$

toric variety

gives dual degeneration

$$Y \times T$$

if this is a singular point then it stops singular in the degeneration

b/c it's trivial

$Y =$ toric variety w/
 momentum polytope B

The point was that before we only got singularities from $\check{Z}$; here there can be singularities coming from $\partial \check{B}$ too. if $\check{B}$ at $v \in \partial \check{B}$ is not standard simplicial over $\mathbb{Z}$ then there is a singularity.

To get a LG model, should start from some kind of dual to this picture. In our setting, duality (or, indeed, mirror symmetry) means the discrete Legendre transform.

Basic duality

take cone over Σ defined by $\mathcal{C}$, dualize the cone and cut at height 1

$$\left(\begin{array}{c} \Sigma \\ \text{fan} \end{array}, \begin{array}{c} \mathcal{C} \\ \text{strictly} \\ \text{convex} \\ \text{PL function} \end{array} \right)$$

bijection
~~diff~~

$$\mathbb{Z}\text{-polytopes}$$

$$\mathcal{M}_{\mathbb{R}}$$

modulo affine bransf.
and affine functions

modulo affine
transformations

$$\check{P} = \{0\}$$



$$(\Sigma, \mathcal{C})$$

$$(\Sigma_v, \mathcal{C}_v)$$



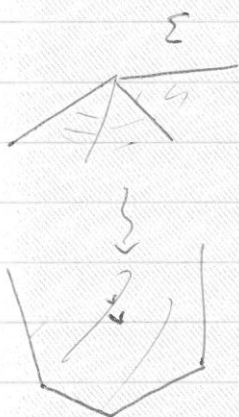
$$\check{V} \rightsquigarrow \mathcal{P}$$

this is
a polytope

polyhedral
decomp.

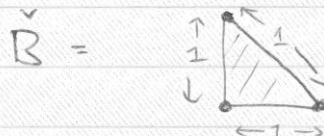


$$(B, P, \mathcal{C}) = \text{DLT}(\check{D}, \check{P}, \check{\mathcal{C}})$$



example

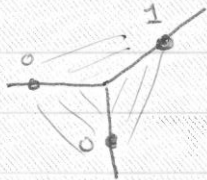
$$(\check{X}, \check{D}) = (P^2, \check{D}) \times T$$



trivial
degeneration
 $\check{P}$ = usual polyhedral
decomposition

dual

$$(B, P, \mathcal{C})$$



there are
values not slope changes

$$B = \mathbb{R}^2$$

nb B closed

$\check{B}$ closed

so CY can work too

How to get the Hori-Vafa mirror here?

B is unbold

What is the family here?
it gives the total space of the Hori-Vafa mirror

what is the family here?
(associated to (θ, φ, D))

$$\mathcal{X} = (x_0, x_1, x_2 - t) \\ \subset \mathbb{C}^3 \times \mathbb{T}$$

Superpotential:

on X_0 , $W_0 =$ unique function defined by

$$\bigcup_{\sigma \in \mathcal{P}} X_\sigma$$

$$W_0|_{X_\sigma} = \begin{cases} 0 & \text{if } \sigma \text{ is bold} \\ \sum_{\omega \in \sigma} z^{m_\omega} & \text{if } \sigma \text{ is unbold edge} \end{cases}$$

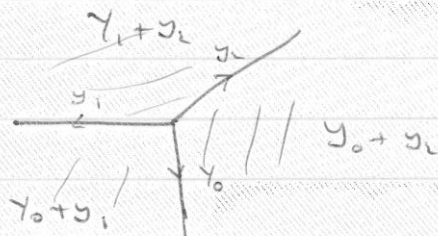
Issue hidden here: does the reconstruction theorem work for non-compact B ?

Yes in dim 2; ~~if~~ probably yes in general but may need extra hypotheses

of course this is true if B is compact

where $m_\omega =$ primitive vector determined by unbold rays

For us: $X_0 = 3$ copies of $\mathbb{C}^1$



Content of the soon-to-be-released paper is ~~that~~ ^{how} one can extend this over the whole family

Now it is desirable for the ~~linear~~ superpotential to be proper, and one only needs to check this on the central fiber.

This has a nice interpretation both in terms of:

- * affine / log geometry of $\tilde{B}$ and $\tilde{D}$
- * Fano geometry

Propⁿ: W_0 is proper $\Leftrightarrow$ all unbdd edges ^{of $\tilde{B}$} are parallel (in each cell) 

so it makes sense to run their program on the boundary of $\tilde{B}$

↑
anticanonical hypersurface = CY inside Fano

$\Leftrightarrow$ the boundary of $\tilde{D}$ is totally geodesic

$\Leftrightarrow (\partial \tilde{B}, \partial \tilde{D}, \tilde{c}|_{\partial \tilde{D}})$ is the intersection complex of the induced degeneration $\tilde{D} \rightarrow T$

$\Leftrightarrow \tilde{D}_\eta$ is irreducible

(so irreducible general fiber is minor to proper superpotential)

Fano $\Leftrightarrow$ LG

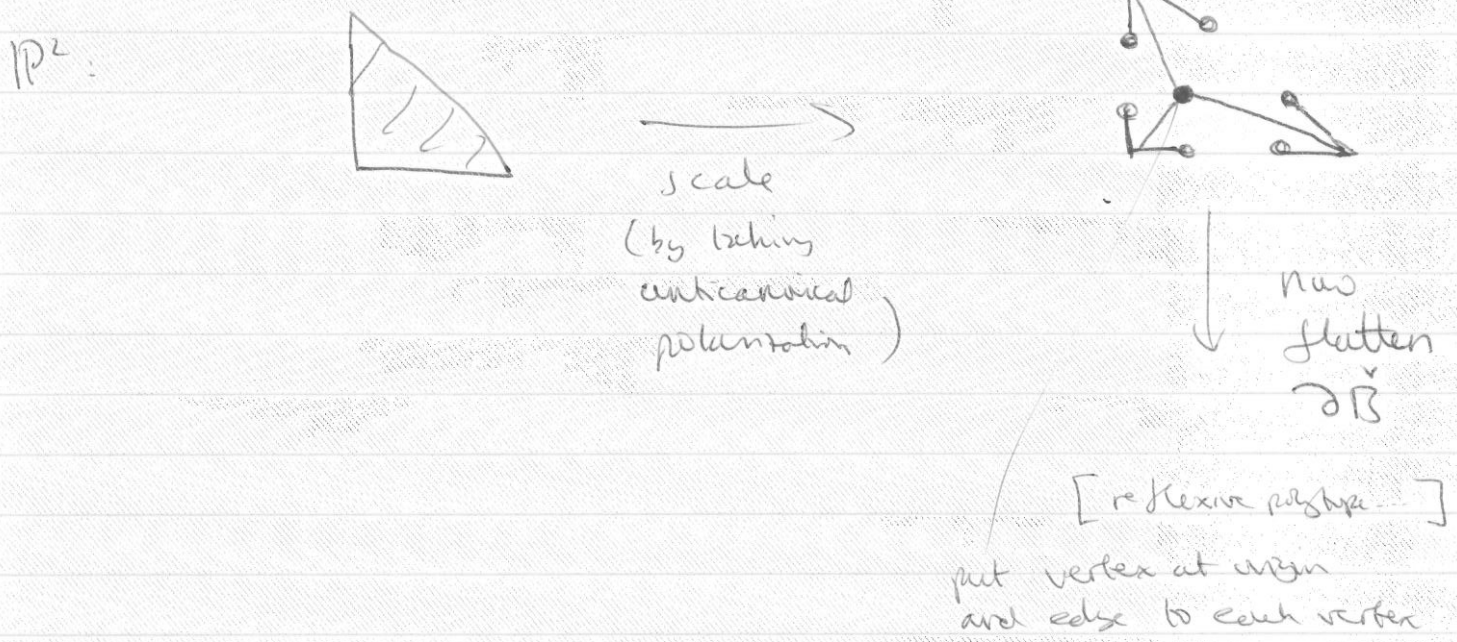
Siebert: for me, ~~linear~~ minor symmetry starts with a toric degeneration with the additional

assumption that the general fiber is irreducible.

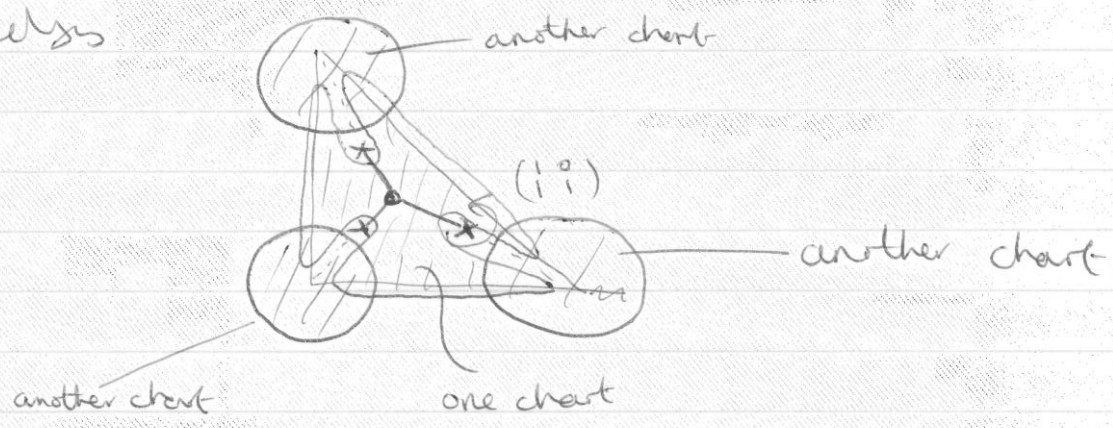
So what to do for $\mathbb{P}^2$?

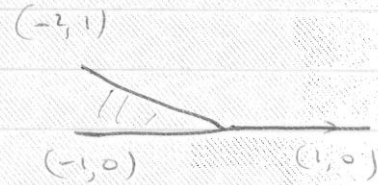
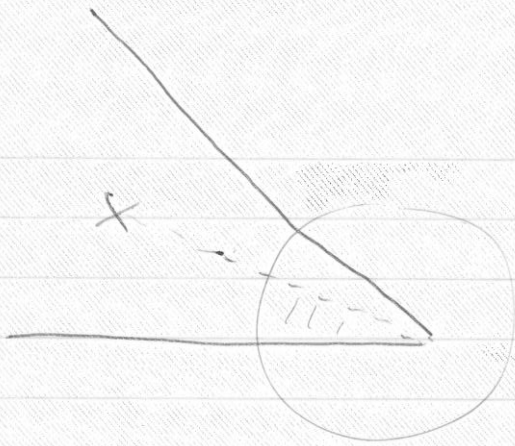
geometrically could take $(\mathbb{P}^2, \mathcal{D})$ and then degenerate the elliptic curve rather than taking the product (= trivial) degeneration.

But instead start on the ~~other~~ affine-geom. side



now put four-fold singularities on each of these added edges





↑
both cones here
are standard

↑
both cones
here
are standard

↔
so 3! chart
map here

How to compute?

Need the notion of broken lines

Not

What is the dual

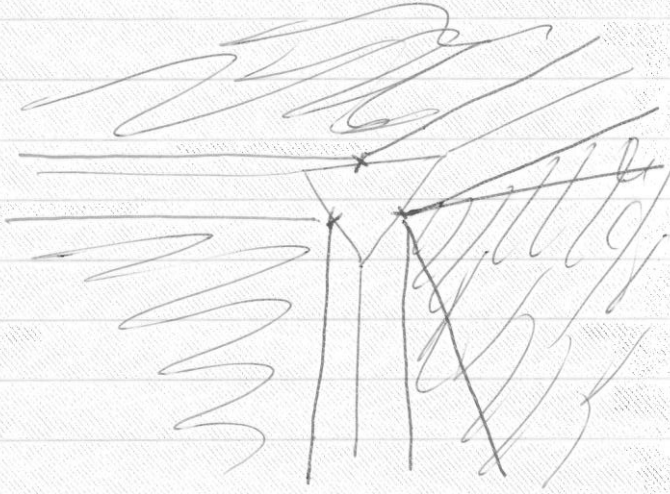
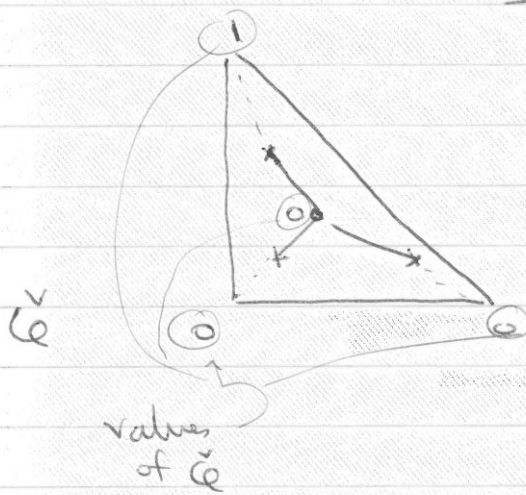
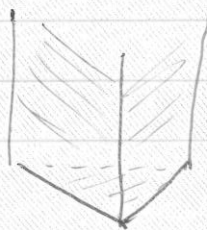


chart is
not a
global
chart

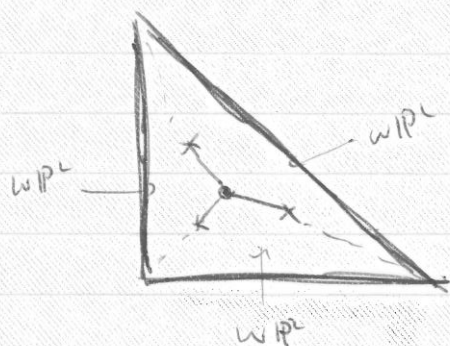
but
 $\mathbb{R}^2$ - shaded
is a
fundamental
domain

} think



3D picture

$\check{X}$ is $\mathbb{P}^2$ degenerates to 3 $\mathbb{W}\mathbb{P}^2$'s

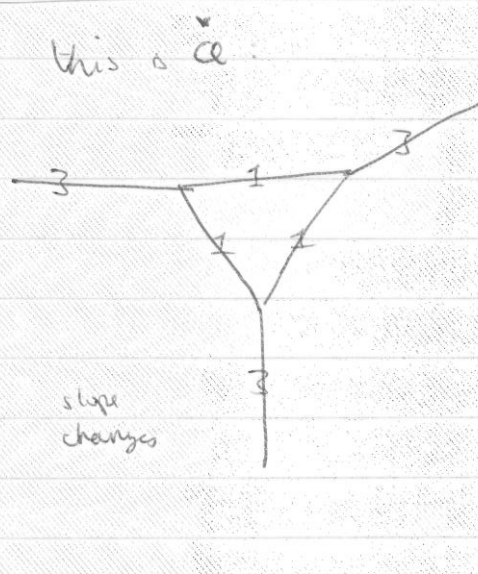


with the elliptic curve ($=\infty$)
getting smooshed into the
indicated divisor

Now consider the dual $\check{X}$

$\check{X}$ is constructed as

$$\lim_{\rightarrow} \check{X}_k \downarrow \text{spec } \mathbb{C}[t]/(t^{k+1})$$



Want to construct $\check{X}_k$ and ω_k (a function on $\check{X}_k$)

Ultimately we want $(\check{X}, \omega)$ with $\omega \in \mathbb{C}_X$

Want to convince you that when we run this, at
least for reflexive polytopes, you get Hori-Vafa
on some $(\mathbb{C}^*)^k \times T$ but canonically
compactified in some way.

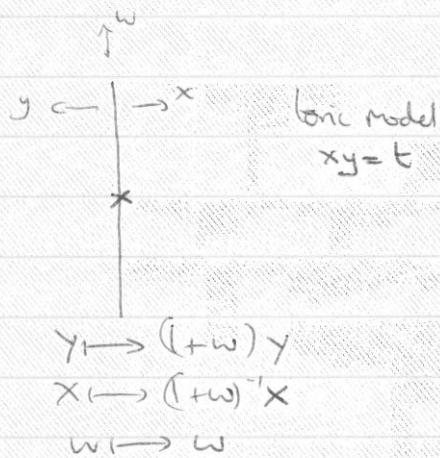
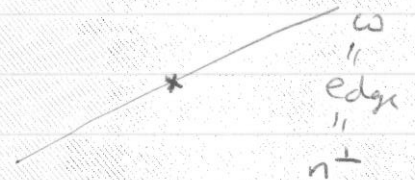
The algorithm works by writing down toric models
coming from (B, P, ϕ) . B/c of monodromy, the
affine toric suggestions ~~do~~ do not patch; but one

can correct this order-by-order by adding corrections ("quantum corrections") emanating from the singular points.

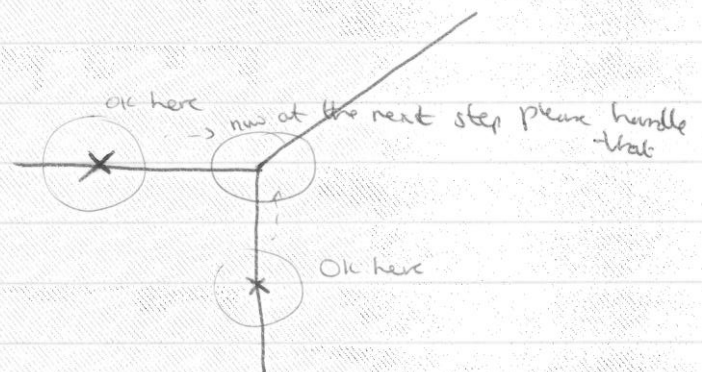
construction of $\mathcal{X}_k$ corrects the gluing along (or near) singular points ($= \Delta$) by transformations (= biholomorphisms) of the form

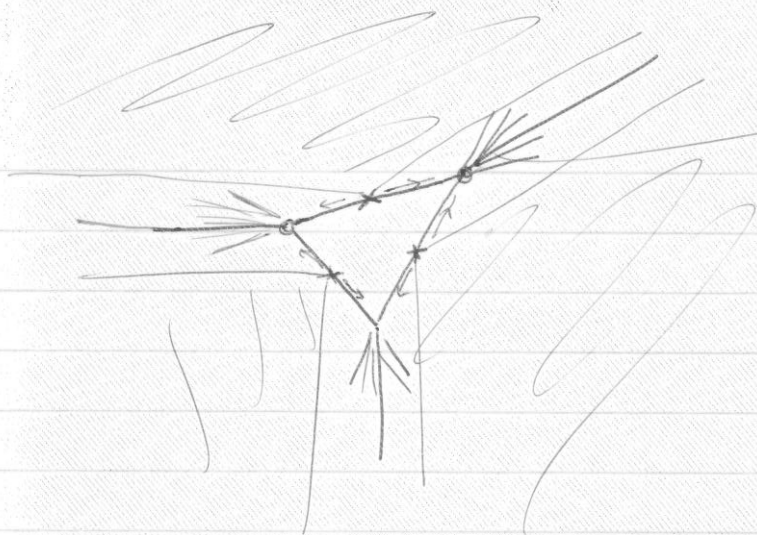
$$z^m \mapsto (1 + f_\omega)^{-\langle n, m \rangle} z^m \quad \text{where}$$

$m=2$ throughout



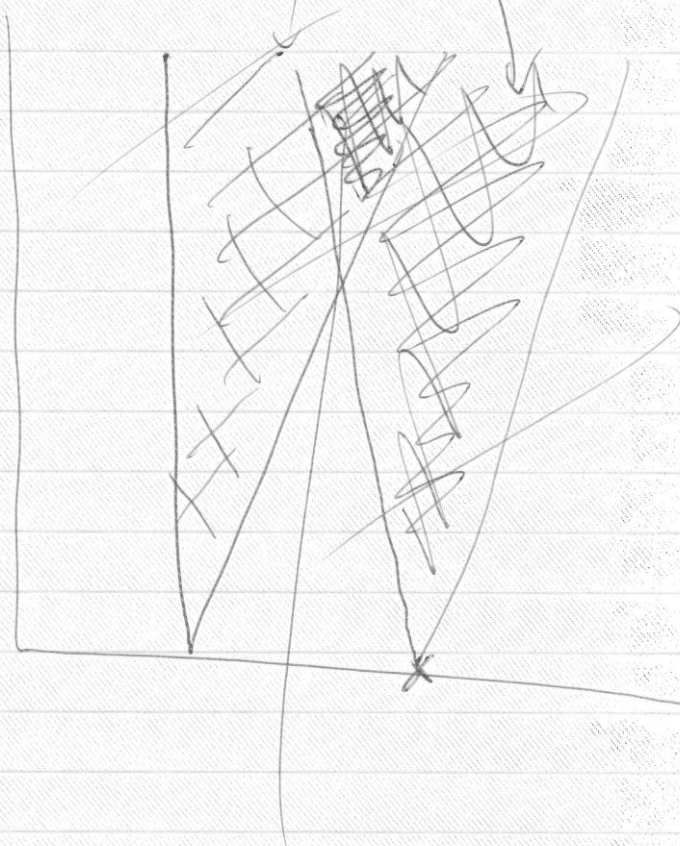
this all glues nicely
but there will be some
problems when different
corrections come together:





"scattering" to make
locally consistent at the vertex

scattering region



the regions overlap

unwrap the
cylinder:

$$(\mathbb{I} \times \mathbb{R}_{\geq 0}) / \sim$$

so we get scattering of scattering of scattering etc.

and there is somehow no hope to control the final answer.
But note that there is no scattering into the
polytope $\rightarrow$ "Huh Vafa mum there survives"

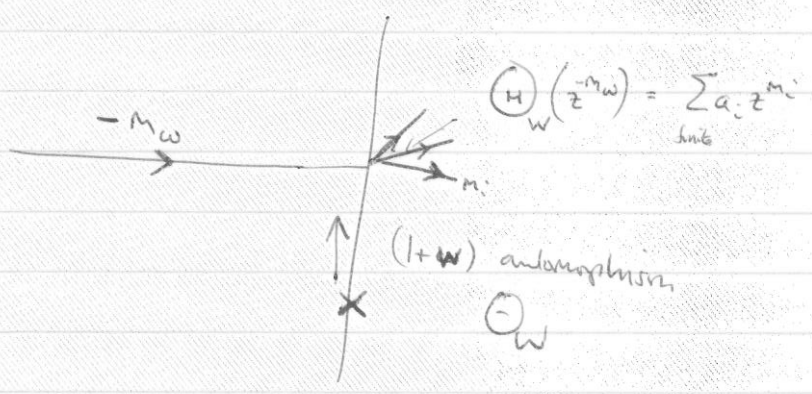
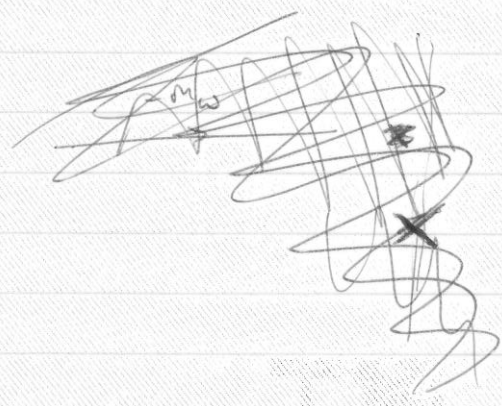
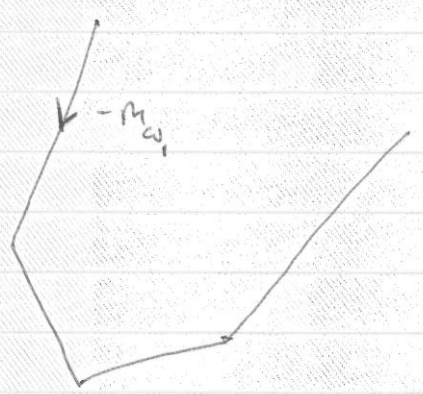
But need now to discuss the construction of W_k
 (so far we have only talked about X_k)

This is due to Mark Gross (at least the essential idea).

For each unbd edge

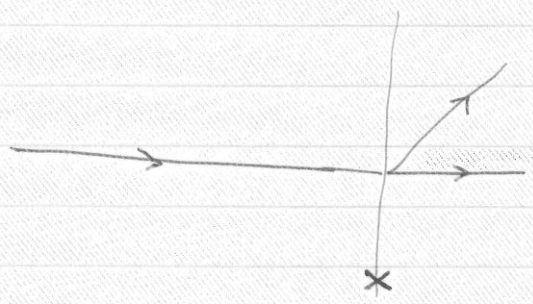
(recall $W^0|_{X_e} = \sum_{\text{wcb unbd edge}} z^{m_w}$)

"shoot" $-m_{w_i}$ in from infinity and it interacts with all the scatterings as follows



so for us

$$\Theta_W(z^{-n_w}) = z^{-n_w} + w z^{-n_w}$$

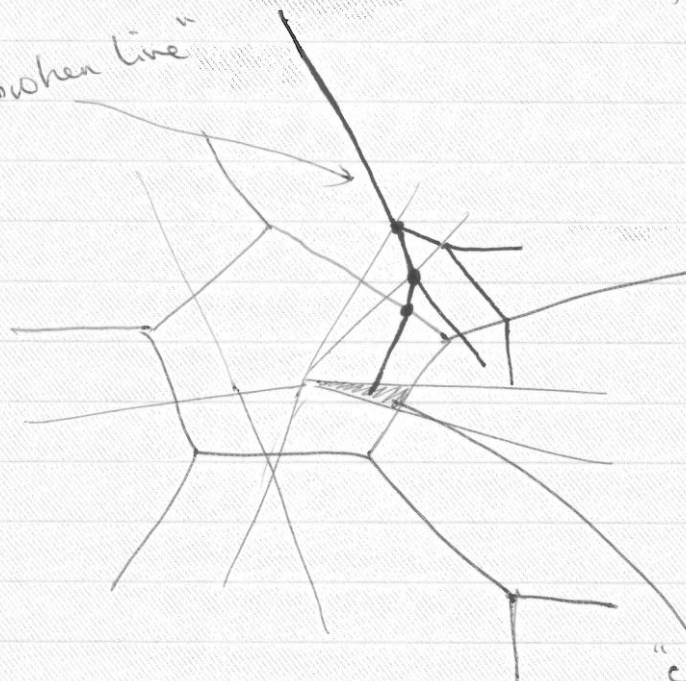


and scatter and scatter
 and scatter etc

To construct $W_k \in \mathcal{O}(X_k)$

finite scattering diagram
is necessary to construct X_k

"broken line"



scattering diagram
for X_k

"chamber" u gives a local
patch of X_k

local patch
 $V(u)$, say

chambers are glued by the
automorphism corresponding
to the scattering rays

So we will construct W_k on a chamber and then
show that it is compatible with gluing.

To construct $W_k|_{V(u)}$

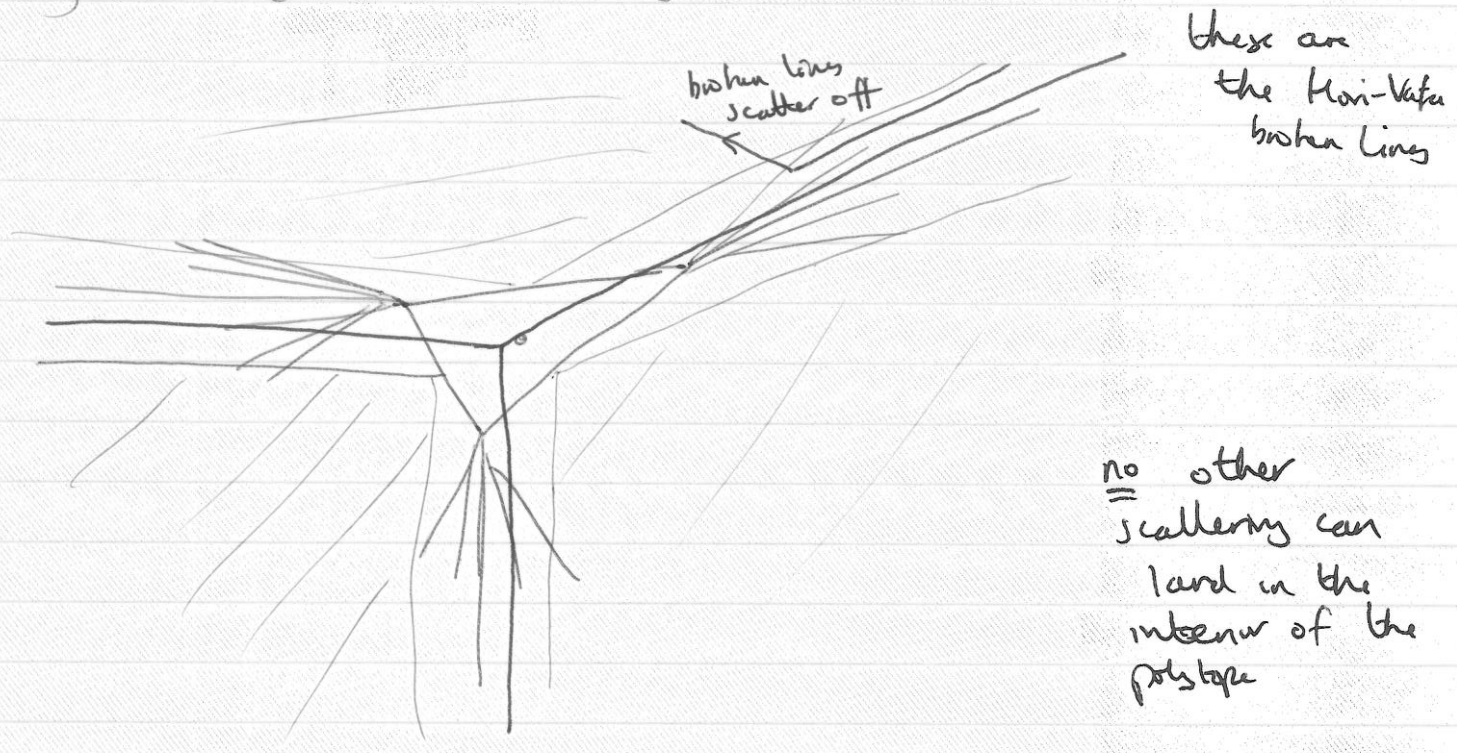
look at how unbold directions can scatter into your
chamber u

$$W^k|_{V(u)} = \sum_{\text{broken lines } \beta} a_\beta z^{m_\beta}$$

Th^m : $W^k|_{V(u)}$ glue

For IP^2 , there are no broken lines except for the ones Mori-Vafa suggests.

This is true for many reflexive polytopes, but not the ones corresponding to singular toric varieties.



So for P^2 (and for any smooth toric surface)

Mori-Vafa is also ~~just~~ the result of this mirror construction.

