Split notes - 1.

SERGEY GALKIN

Abstract. This is a part of my talk *Non-commutative mirror symmetry* on July 13, 2011 at *HMS and CT* workshop in Split. I'll discuss natural importance of positivity and convexity.

1. Laurent polynomials and random walks.

Let M be n-dimensional lattice $\mathbb{Z}^n$, $t_1, \ldots, t_n$ be coordinates on affine space $\mathbb{A}^n$ and $x_i = e^{t_i}$ be coordinates on the torus G_m^n. Note that if t_i are real then x_i are real and positive.

Consider a Laurent polynomial W with real positive coefficients.

$$W = \sum a_m x^m = \sum a_m e^{tm} \tag{1}$$

Restriction of W to real positive values of x_i is a real positive function on $\mathbb{R}^n$.

Denote $A = \sum a_m$. Note that $A = W(1, \ldots, 1)$ i.e. A is a particular value of W.

Assume additionally that $0 \in M$ lies in the interior of Newton polytope of W.

Lemma 2. *Function W has a unique critical value on $\mathbb{R}^n$ and this critical value is a global minimum.*

Proof. Since 0 is contained in the interior of the Newton polytope for every direction $t \to \infty$ one of the monomials of W also goes to $+\infty$. Since all coefficients are positive W goes to $+\infty$ as well. This implies W has at least one minimum on $\mathbb{R}^n$.

As pointed out by Mikhalkin, W is a convex function in coordinates t_i: each monomial e^{tm} is convex, so sum of monomials with positive coefficients is also convex. Since convex functions has at most one critical point we are done. □

Let W_{min} be equal to the minimal value of W on real positive part.

Proposition 3. *For $W_0 > W_{min}$ the fibers $W^{-1}(W_0)$ are diffeomorphic to $(n-1)$-dimensional sphere S^{n-1}.*

Proof. Once we know the lemma above apply the standard argument from Morse theory. □

We point that uniqueness of real positive critical point will also follow from the arguments below, where we give an estimate of the respective critical value W_{min}.

Consider n-cycle $\Gamma = \{|x_i| = 1\}$ and a holomorphic volume form $\omega = \frac{1}{(2\pi i)^n} \prod \frac{dx_i}{x_i} = \frac{1}{(2\pi i)^n} \prod dt_i$ on n-dimensional complex torus $(\mathbb{C}^*)^n$. For Laurent polynomial F denote by $Tr(F)$ its constant term $\int_\Gamma F$.

Define G-function and $\hat{G}$-function as follows:

Date: August 3, 2011.

$$\hat{G}_W = \int_\Gamma \frac{\omega}{1 - tW} = \sum_{k \geqslant 0} Tr(W^k) t^k \tag{4}$$

$$G_W = \int_\Gamma e^{tW} \omega = \sum_{k \geqslant 0} Tr(W^k) \frac{t^k}{k!} \tag{5}$$

Clearly $\hat{G}_W$ is Laplace transform of G_W. Moreover, as we'll seee soon G_W has infinite radius of convergence while radius of convergence of $\hat{G}_W$ is finite.

Lemma 6 (Dutch trick)**.** *Function $\hat{G}_W$ is a period for the family $1 - tW = 0$ of hypersurfaces in the torus $(\mathbb{C}^*)^n$.*

Denote by $R = R_W$ radius of convergence of $\hat{G}_W$, and let $T = T_W = \frac{1}{R}$.
Lemma 6 implies that T is the maximal absolute value of all critical values of W.
By Cauchy-Hadamard formula $T = \lim_{k \to \infty} Tr(W^k)^{\frac{1}{k}}$.

Proposition 7. *We have a bound $T \leqslant A$, and more generally $T \leqslant W(\alpha)$ for any real positive α.*

Proof. "*Probability is bounded by* 1": since W is a Laurent polynomial with real positive coefficients, W^k is also a Laurent polynomial with real positive coefficients, so W^k equals to sum of positive monomials which are positive evaluated at any positive point, and $Tr(W^k)$ is one particular term, so it is bounded by the sum. □

Proposition above shows the inequality

$$T \leqslant W_{min} \tag{8}$$

On the other hand T equals to maximal absolute value of all complex critical values of W. In particular this implies there are no other critical points of W on $\mathbb{R}^n$ except for the global minimum.

To be continued...

2. Invariant T of Fano varieties and mirror symmetry.

3. Reconstructions.

4. Laurent phenomenon.

5. Three incarnations and three levels.

6. Quantized and non-commutative random walks.

References