Split notes - 1,2,3.

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Abstract. This text has non-empty intersection with my talk *Non-commutative mirror symmetry* on July 13, 2011 at *HMS and CT* workshop in Split. I'll discuss natural importance of positivity and convexity. Obro vector and characterization of spectra of maximal radius I added in October.

1. Laurent polynomials and random walks.

Let N be a lattice with dual $M = \mathrm{Hom}(M, \mathbb{Z})$.Let u be coordinates on affine space $M \otimes_{\mathbb{Z}} \mathbb{A}^1$ and $x = e^u$ be coordinates on the torus $T = \mathrm{Hom}(N, \mathbf{G}_m) = M \otimes_{\mathbb{Z}} \mathbf{G}_m$. Note that map exp is an isomorphism between domains where all u are real and all x are real positive, further we denote this domain by V.

Consider a Laurent polynomial W with complex coefficients

$$W = \sum_{l \in N} a_l x^l = \sum_{l \in N} a_l e^{un} \tag{1}$$

Denote $A = \sum |a_l|$. Note that all coefficients a_l are real and non-negative $\iff$ $A = W(1)$. In this case we can consider restriction of W to V as a function of real positive argument x and A as its particular value. Furthermore in case $A = 1$ one may interpret Laurent polynomial W as a random walk in the lattice M: coefficient a_l is probability to go in direction m and each next step is independent of the past. In case $A \neq 1$ function W is simply a rescale of the probabilistic one.

Assume additionally that the origin $0 \in N$ lies in the interior of Newton polytope of W (so random walker has some chance to come back to the origin).

Remark 2. I consider properties of positivity and convexity (and their corollaries discussed below) as an Archimedean counterpart to p-crystal properties.

Definition 3. We say that point $x_0 \in (\mathbb{C}^*)^n$ is a usual *critical point* of W if $dW|_{x=x_0} = 0$. We say that $c \in \mathbb{C}$ is a usual *critical value* of W if $c = W(x_0)$ for some usual critical point x_0.

Lemma 4. *Function W has a unique critical point on V and it is the global minimum.*

Proof. Since 0 is contained in the interior of the Newton polytope for every direction $|u| \to \infty$ one of the monomials of W also goes to $+\infty$. Since all coefficients are positive W goes to $+\infty$ as well. This implies W has at least one minimum on V.

Note that W is a convex function in coordinates t_i: each monomial e^{tm} is convex, so sum of monomials with positive coefficients is also convex. Since convex functions has at most one critical point we are done. □

Let W_{min} be the minimum of W on real positive part i.e. the value of W at the unique critical point with real positive coordinates.

Proposition 5. *For $W_0 > W_{min}$ the fibers $W^{-1}(W_0) \subset V$ are diffeomorphic to $(n-1)$-dimensional sphere S^{n-1}.*

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Proof. Once we know the lemma above apply the standard argument from Morse theory. □

We point that uniqueness of real positive critical point will also follow from the arguments below, where we give an estimate of the respective critical value W_{min}.

Consider n-cycle $\Gamma = \{|x| = 1\}$ and a holomorphic volume form $\omega = \frac{1}{(2\pi i)^n} \prod \frac{dx_i}{x_i} = \frac{1}{(2\pi i)^n} \prod dt_i$ on n-dimensional complex torus $(\mathbb{C}^*)^n$.

Definition 6. For Laurent polynomial W denote by $Tr(W)$ its constant term $\int_\Gamma W$. Let $Traces(W)$ be the set of natural numbers k such that $Tr(W^k) \neq 0$ ($k = 0$ is included), and *index* $r(W)$ be the greatest common divisor of all elements in $Traces(W)$. Define G-function and $\hat{G}$-function as (exponential) generating function for $Tr(W^k)$:

$$\hat{G}_W = \sum_{k \geqslant 0} Tr(W^k) t^k = \int_\Gamma \frac{\omega}{1 - tW} \tag{7}$$

$$G_W = \sum_{k \geqslant 0} Tr(W^k) \frac{t^k}{k!} = \int_\Gamma e^{tW} \omega \tag{8}$$

Number $Tr(W^k)$ can be interpreted as probability to be back at the origin after k independent steps, so function $\hat{G}_W$ is generating function for these probabilities.

Proposition 9. *For positive W_1, W_2 we have $Tr(W_1 \cdot W_2) \geqslant Tr(W_1) \cdot Tr(W_2)$.*

Corollary 10. *The set $Traces(W)$ is an additive monoid: if $a, b \in Traces(W)$ then $(a + b) \in Traces(W)$. This implies that there is some number n_0 such that for all $n \geqslant n_0$ number $r(W)n$ belongs to $Traces(W)$.*

Denote by $R = R_W$ radius of convergence of $\hat{G}_W$, and define invariant $T = T_W = T(W) = \frac{1}{R}$.

Lemma 11. *Power series $G_W(t)$ exponentially converge everywhere on complex line and power series $\hat{G}_W(t)$ have positive radius of convergence R and $T \leqslant \sum |a_l|$.*

Proof. In case all coefficients a_l are real non-negative one can use "*probability is bounded by* 1" argument: since W is a Laurent polynomial with real positive coefficients, W^k is also a Laurent polynomial with real positive coefficients, so W^k equals to sum of positive monomials which are positive evaluated at any positive point, and $Tr(W^k)$ is one particular term, so it is bounded by the sum which is A^k. This implies $T = \lim_{k\to\infty} Tr(W^k)^{\frac{1}{k}} \leqslant \lim_{k\to\infty} (A^k)^{\frac{1}{k}} = A$. Case where some coefficients have non-zero argument can be absolutely bounded by the positive case. Exponential convergence of G_W immideately follows from convergence of $\hat{G}_W$. □

Lemma 12. *If $\hat{G}_W = \sum_{n \geqslant 0} g_n t^n$ then $\hat{G}_{W^k} = \sum_{n \geqslant 0} g_{kn} t^n$.*

Clearly $\hat{G}_W$ is Laplace transform of G_W.

Lemma 13 (Dutch trick)**.** *Function $\hat{G}_W$ is a period for the family $1 - tW = 0$ of hypersurfaces in the torus $(\mathbb{C}^*)^n$.*

Definition 14. Define spectrum of W as the set of inverses of critical points of $\hat{G}_W$.

Lemma 15. *Spectrum of Laurent polynomial W in the sense of definition 14 contains all usual critical values of W in the sense of definition 3.*

Proof. □

Lemma 13 implies that T is the maximal absolute value of all critical values of W (in a broad sense - including values at critical points outside the torus $(\mathbb{C}^*)^n$).

By Cauchy-Hadamard formula $T = \lim_{k\to\infty} Tr(W^k)^{\frac{1}{k}}$.

Proposition 16. *There is an upper bound $T \leqslant W(\alpha)$ for any real positive α.*

Proof. Lemma 11 implies $T \leqslant A = W(1)$. On the other hand for any real positive α Laurent polynomial $W'(x) = W(\alpha x)$ has the same $\hat{G}$-function (since Jacobian of the coordinate change equals one). However $W'(1) = W(\alpha)$. So same argument as above shows that $T \leqslant W(\alpha)$ for any real positive n-tuple α. □

Proposition above shows the inequality

$$T \leqslant W_{min} \tag{17}$$

On the other hand T equals to maximal absolute value of all complex critical values of W. In particular this implies there are no other critical points of W on $\mathbb{R}^n$ except for the global minimum.

This implies that positive Laurent polynomials have a unique "canonical" real positive coordinates – namely coordinates where the unique real positive critical point is $(1, 1, \dots, 1)$. Indeed this fixes a "translational" symmetry of the torus ($t_i \to t_i + b_i$ or $x_i \to x_i \times a_i$), however there is still some "rotational" symmetry possible (which preserves the set of coefficients of W), and in fact these symmetries can be further exploited (see below).

Definition 18. In case a critical point of positive Laurent polynomial W is $t_i = 0$ we say that W is a *balanced* Laurent polynomial and t_i are balanced coordinated.

Definition 19. For Laurent polynomial $W = \sum a_n x^m$ define its *Obro vector* as $Obro(W) = \sum_{m\in M} a_n \cdot m$. Probabilistic interpretation of Obro vector is the average drift of random walker per one step. And the third interpretation: Obro vector is proportional to the centre of mass of a system of point particles positioned at the lattice points m with respective masses a_n.

Proposition 20. *Positive Laurent polynomial W is balanced $\iff$ its Obro vector vanishes $Obro(W) = 0 \in M \iff T(W) = W(1)$.*

Proof. Note that Obro vector equals to de Rham differential of W evaluated at $x = 1$ under natural isomorphism $T^*_{(1,\dots,1)}\mathbf{G}_m(\mathbb{R}) \simeq M \otimes \mathbb{R} : Obro(W) = dW|_{x=1}$. □

This immediately implies that

Proposition 21. *All balanced (maybe non-positive) Laurent polynomials form a subalgebra in algebra of all Laurent polynomials.*

Proof. Indeed the map $W \to dW|_{x=1}$ is linear and product of balanced polynomials is balanced by Leibniz rule. In fact balanced polynomials satisfying $W(1) = 0$ form an ideal in the ring of balanced polynomials and this ideal is the square of the ideal of Laurent polynomials vanishing at 1. □

Corollary 22. *Map $W \to T(W)$ restricted to balanced positive Laurent polynomials coincides with homomorphism of rings $W \to W(1)$. So if W_1 and W_2 are balanced Laurent polynomials and α_1, α_2 are positive numbers then $T(\alpha_1 W_1 + \alpha_2 W_2) = \alpha_1 T(W_1) + \alpha_2 T(W_2)$ and $T(W_1 W_2) = T(W_1)\cdot T(W_2)$.*

Definition 23. Define *index* $r(W)$ as the greatest common divisor of natural numbers n such that $Tr(W^k) \neq 0$ (index of constant function is defined to be ∞).

Remark 24. Index $r(W)$ can be also characterized as the greatest number r such that $\hat{G}_W(e^{\frac{2\pi i}{r}}t) = \hat{G}_W(t)$. or in other words $\hat{G}_W(t) = \hat{G}_{W^r}(t^r)$. From wandering drunkard's point of view this means that is return to the origin is possible only in number of steps divisible by r.

Lemma 25. *Indices of W and its powers are related as follows: $r(W^k) = \frac{r(W)}{gcd(k,r(W))}$. In particular $r(W^{r(W)}) = 1$.*

Theorem 26. *Complex number T' such that $|T'| = T_W$ is an element of spectrum of W $\iff$ $T'^r = T^r$ i.e. $T' = T \cdot e^{\frac{2\pi ip}{r(W)}}$ for some integer p.*

Proof. Since the spectrum is invariant of $\hat{G}_X$ the the inclusion statement follows from definition of index. Let us prove that other points on the circle of radius T do not lie in the spectrum. Lemmas 25 and **??** applied to W and $W^{r(W)}$ reduces the problem to the case $r(W) = 1$.
□

To be continued...

2. Invariant T of Fano varieties and mirror symmetry.

For Fano variety X denote by J_X its Givental's J-function $(ev_1)_* \frac{z}{z-\psi_1}$ restricted to anticanonical direction $\mathbf{t} = tc_1(X)$ and $z = 1$. Consider $G_X = \int_{[X]} J_X \cup [pt]$ and its Fourier-Laplace transform $\hat{G}_X$.

Definition 27 (See [2, 5])**.** Define *spectra Spectra*(X) [1] as the collection of inverses of all critical points of the function $\hat{G}_X$. Equivalently spectra of Fano variety is the collection of roots of its *quantum characteristic polynomial* (characteristic polynomial of the operator of quantum multiplication by $c_1(X)$).

Definition 28. Define $T(X)$ as inverse of radius convergence of $\hat{G}_X$. Equivalently, $T(X)$ is maximal absolute value of elements in the spectrum of X.

Definition 29. We say that Laurent polynonial W *reflects* Fano variety X if $G_W = G_X$ (or, equivalently, $\hat{G}_W = \hat{G}_X$).
[2]

Example 30. Let X be a smooth toric Fano variety and v_i — primitive generators on the rays of its fan. Then Laurent polynomial $W = \sum x^{v_i}$ reflects X by results of Givental [4]. Further we call this function W as the standard reflection for toric Fano variety X.

Question 31. *We are going to address the following questions:*

1. *What are the possible values of number $T(X)$ for Fano varieties X.*
2. *In particular, what are the bounds?*
3. *How they depend on dimension?*
4. *What are the values of $T(X)$ for toric Fano varieties and how they differ from generic?*

Theorem 32. *For smooth toric Fano variety X there is an upper bound*

$$T(X) \leqslant \dim X + \rho(X) \leqslant 3 \dim X.$$

[1] Anticanonical spectrum in notations of [5].

[2] In such situation Przyjalkowski says that W is *very weak Landau–Ginzburg model* (mirror dual) to X.

Proof. Consider the standard reflection W from example 30. We have $T(X) = T(W)$, by 16 $T(W) \leqslant W(1)$, finally $W(1)$ equals to number of vertices in the fan of smooth toric variety and this number equals $\dim X + \rho(X)$.

Inequality $\rho(X) \leqslant 2\dim X$ is proven in [1], and we'll reproduce much simpler proof of Obro from [7]. Consider *special facet* F — any facet whose cone contains Obro vector, and let f_F be a linear function on M that equals 1 on F. Note that $\sum_{v\in Vertices(X)} f_F(v) = f_F(Obro(X)) \geqslant 0$. Any vertex v of X falls in one of three categories by sign of $f_F(v)$. If $f_F(v) > 0$ then v is one of d vertices of facet F, so $f_F(v) = 1$ and $\sum_{v:f_F(v)\geqslant 0} f_F(v) = d$. Number of v with $f_F(v) < 0$ is bounded by d since $f_F(Obro(X)) \geqslant 0$. Number of vertices v with $f_F(v) = 0$ is also bounded by d due to combinatorial reasons. Altogether number of vertices is bounded by $d + d + d = 3d$. □

3. Reconstructions.

Definition 33. *Ano variety* Y is a complement in a smooth Fano variety F to its smooth anti-canonical hypersurface $A \in |-K_F|$: $Y \simeq F - A$.

Lemma 34. *If* Y *is Ano variety then* $H^0(Y, \mathcal{O}_Y^*) = \mathbb{C}$.

Proof. Any function $f \in H^0(Y, \mathcal{O}_Y^*)$ can be considered as a rational function on F. Since functions f and $\frac{1}{f}$ are regular on Y their divisors of poles should be supported at A. However divisor of poles of $\frac{1}{f}$ is divisor of zeroes of f, so since A is irreducible function f doesn't have any poles or zeroes, so its a regular function on projective variety F that is a constant. □

Theorem 35. *The respective Fano variety* F *and its anti-canonical Calabi–Yau section* A *can be uniquely reconstructed from Ano variety* Y.

Proof. □

Definition 36. Smooth variety U is called *Ona variety* if its canonical line bundle is trivial $K_U = \mathcal{O}_U$ and it has a flat projective map to $w : U \to \mathbb{A}^1$ with connected fibers, and generic fiber is smooth.

Lemma 37. *If* Y *is Ona variety then* $H^0(Y, \mathcal{O}_Y) = \mathbb{C}[w]$.

Theorem 38. *The respective map* $w : Y \to \mathbb{A}^1$ *from Ona variety to* $\mathbb{A}^1$ *can be uniquely (up to automorphisms of* $\mathbb{A}^1$*) reconstructed from Ona variety* U.

Proof. □

4. Laurent phenomenon.

5. Three incarnations and three levels.

All this story has 3 incarnations: C (for commutative or classical), Q (for quantized) and NC (non-commutative [6]).

5.1. **Potentials.** C-potentials (of the usual commutative theory) are just usual Laurent polynomials. We may consider Laurent polynomial as an element of a group ring of a free abelian group.

Q-potentials (or *quantized Laurent polynomials*) are elements of the quantum torus i.e. group ring of Heisenberg group. Sometimes it is more convenient to work with a double central extension of Heisenberg group. Let q be the generator of the center of Heisenberg group.

Finally, NC-potentials are the noncommutative Laurent polynomials e.g. elements of a group ring of a free group.

5.2. G**-functions.** One defines the *trace* of a potential $W = \sum c_g x^g$ as its constant term $Tr(W) = c_1$ (where 1 is the identity element in the respective group).

In quantized setup one can also define the *central trace* as the sum of all central monomials $Tr(\sum c_g x^g) = \sum_{n \in \mathbb{Z}} c_{q^n} q^n$.

The name trace is partially motivated by the fact it vanishes on commutators i.e. $Tr(ab) = Tr(ba)$.

G-functions are defined as various generating functions for traces of powers of W, and can also be thought as generalized characteristic polynomials. As well these generating series count the probabilities to come in n steps to the origin for random walker on a respective group.

5.3. **Coordinate change formulae.** Assume we have some coordinate transformation (automorphism of (skew-)field of fractions of group ring of group G). Additionally assume a Laurent phenomenon: some potential W is mapped into another potential W'.

Under which conditions the G-functions are preserved i.e. traces of powers of W remain the same.

The commutative case is served by a coordinate change formula in integral: the Jacobian of the transformation is identity $\iff$ the holomorphic volume form ω on torus is mapped into itself.

It is a delightful gift of quantization that in Q case any coordinate change that preserves q is fine.

I don't know under what conditions (if any) G-functions are preserved in NC setting.

6. Quantized and non-commutative random walks.

7. Appendix: Futaki-Mabuchi polynomial, degenerations, stabilities and Kahler-Einstein metrics.

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