Small toric degenerations of Fano threefolds

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Abstract. We show which of the smooth Fano threefolds admit degenerations to toric Fano threefolds with ordinary double points.

By theorem 3 we answer Batyrev's question [2][question 3.9]: *Which of the smooth Fano threefolds admit small toric degenerations?* All applications of this result to mirror symmetry and classification of varieties are to appear in separate sequel papers [3, 4].

1. Statement.

Definition 1. *Deformation* is a flat proper morphism $\pi : \mathcal{X} \to \Delta$, where Δ is a unit disc $\{|t| < 1\}$, and $\mathcal{X}$ is an irreducible complex manifold.

All the deformations we consider are projective (π is a projective morphism over Δ). Denote fibers of π by X_t, and let $i_{t\in\Delta}$ be the inclusion of a fiber $X_t \to \mathcal{X}$. If all fibers $X_{t\neq 0}$ are nonsingular, then the deformation π is called a degeneration of $X_{t\neq 0}$ or a smoothing of X_0. If at least one such morphism π exists, we say that varieties $X_{t\neq 0}$ are *smoothings* of X_0, and X_0 is a *degeneration of* $X_{t\neq 0}$.

For a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ over Δ and $t \in \Delta$ the symbol $\mathcal{F}_t$ stands for the restriction $i_t^*\mathcal{F}$ to the fiber over t. In particular there is a well-defined restriction morphism on Picard groups $i_t^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_t)$.

Definition 2 ([2])**.** Degeneration (or a smoothing) π is *small*, if X_0 has at most Gorenstein terminal singularities (see [12] or [13]), and for all $t \in \Delta$ the restriction $i_t^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_t)$ is an isomorphism.

All 3-dimensional terminal Gorenstein toric singularities are nodes i.e. ordinary double points analytically isomorphic to $(xy = zt) \subset \mathbb{A}^4 = \mathrm{Spec}\,\mathbb{C}[x, y, z, t]$.

Theorem 3. *These and only these families of nontoric smooth Fano 3-folds Y do admit small degenerations to toric Fano threefolds:*

1. 4 *families with* $\mathrm{Pic}(Y) = \mathbb{Z}$*:* Q, V_4, V_5,V_{22}*;*
2. 16 *families* $V_{2.n}$, *for* $n = 12, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32$*;*
3. 16 *families* $V_{3.n}$, *for* $n = 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24$*;*
4. 8 *families* $V_{4.n}$, *for* $n = 1, 2, 3, 4, 5, 6, 7, 8$.

Here $V_{\rho,n}$ is a variety indexed $\rho.n$ in table [6] *of Fano threefolds with Picard number ρ. All these degenerations are listed in section 2.*

Definition 4. *Principal invariants* of smooth Fano threefold X is a set of 5 numbers $(\rho, r, \deg, b, d)$ where $\rho = \mathrm{rk}\,\mathrm{Pic}(X) = \dim H^2(X)$ is *Picard number*, $b = \frac{1}{2}\dim H^3(X)$ is a half of third Betti number, $\deg = (-K_X)^3$ is *(anticanonical) degree*, r is *Fano index* defined in 6 and d is *(anticanonical) discriminant* defined in 7.

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Remark 5. All smooth threefolds Y from 3 satisfy the following conditions: Y is rational (see e.g. [11]), $\deg(Y) \geqslant 20$, $\rho(Y) \leqslant 4$, $b(Y) \leqslant 3$, moreover $b(Y) = 3$ only if Y is $V_{2.12}$ and $b(Y) = 2$ only if Y is V_4 or $V_{2.19}$.

Definition 6. The *index* of a (Gorenstein) Fano variety X is the greatest $r \in \mathbb{Z}$ such that anticanonical divisor class $-K_X$ equals rH for some integer Cartier divisor class H.

Definition 7. Let $H \in \mathrm{Pic}(X)$ be a Cartier divisor an on n-dimensional variety X, and $D_1, \ldots, D_l$ be a base of lattice $H^{2k}(X, \mathbb{Z})/tors$. Define $d^k(X, H)$ as a discriminant of the quadratic form $(D_1, D_2) = (H^{n-2k} \cup D_1 \cup D_2)$ on $H^{2k}(X, \mathbb{Z})/tors$. For a Gorenstein threefold X denote by $d(X) = d^1(X, -K_X)$ the anticanonical discriminant of X. If X is a smooth variety and H is an ample divisor, then hard Lefschetz theorem states that $d^k(X, H)$ is nonzero.

We sketch the proof 3. Consider a toric Fano threefold X with ordinary double points.

(1) [26, 25],[27],[14] There is only a finite number of such X and all these threefolds X are explicitly classified.
(2) [21, 22] Every such threefold X admits a smoothing — a Fano threefold Y.
(3) [28, 29, 23] Principal invariants of Y can be expressed via invariants of X.
(4) [8] Explicit classification of smooth Fano threefolds [5, 6, 7, 8, 9, 10] shows that every family Y is completely determined by its principal invariants.
(5) If some smooth Fano threefold Y admits a degeneration to a nodal toric Fano X, then the pair (Y, X) comes from the steps above.

2. Answer

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
V_{22}	1	22	0	22	(13,9,13)	1
V_4	1	32	2	8	(8,6,6)	1
V_5	1	40	0	10	(7,3,7)	1
Q	1	54	0	6	(5,1,5)	1

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
$V_{2.12}$	2	20	3		(14,12,12)	1
$V_{2.17}$	2	24	1		(12,8,12)	1
$V_{2.19}$	2	26	2		(11,8,10)	1
$V_{2.20}$	2	26	0		(11,6,12)	2
$V_{2.21}$	2	28	0		(10,5,11)	2
$V_{2.21}$	2	28	0		(11,6,12)	1
$V_{2.23}$	2	30	1		(9,5,9)	1
$V_{2.22}$	2	30	0		(10,5,11)	1
$V_{2.22}$	2	30	0	$[-24]$	(9,4,10)	1
$V_{2.24}$	2	30	0	$[-21]$	(9,4,10)	1
$V_{2.25}$	2	32	1		(8,4,8)	1
$V_{2.25}$	2	32	1		(9,5,9)	1
$V_{2.26}$	2	34	0		(10,5,11)	1
$V_{2.26}$	2	34	0		(8,3,9)	1
$V_{2.26}$	2	34	0		(9,4,10)	1

$V_{2.27}$	2	38	0		(7,2,8)	1
$V_{2.27}$	2	38	0		(8,3,9)	2
$V_{2.28}$	2	40	1		(7,3,7)	1
$V_{2.29}$	2	40	0		(7,2,8)	1
$V_{2.29}$	2	40	0		(8,3,9)	1
$V_{2.30}$	2	46	0	$[-12]$	(6,1,7)	1
$V_{2.31}$	2	46	0	$[-13]$	(6,1,7)	1
$V_{2.31}$	2	46	0	$[-13]$	(7,2,8)	1
$V_{2.32}$	2	48	0		(6,1,7)	1
$V_{2.34}$	2	54	0		(6,1,7)	1

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
$V_{3.7}$	3	24	1		(12,7,13)	1
$V_{3.10}$	3	26	0		(11,5,13)	1
$V_{3.11}$	3	28	1		(10,5,11)	1
$V_{3.12}$	3	28	0		(10,4,12)	1
$V_{3.12}$	3	28	0		(11,5,13)	1
$V_{3.13}$	3	30	0		(10,4,12)	2
$V_{3.13}$	3	30	0		(9,3,11)	1
$V_{3.14}$	3	32	1		(8,3,9)	1
$V_{3.15}$	3	32	0		(10,4,12)	1
$V_{3.15}$	3	32	0		(9,3,11)	3
$V_{3.16}$	3	34	0		(8,2,10)	1
$V_{3.16}$	3	34	0		(9,3,11)	1
$V_{3.17}$	3	36	0	[28]	(8,2,10)	2
$V_{3.17}$	3	36	0	[28]	(9,3,11)	1
$V_{3.18}$	3	36	0	[26]	(8,2,10)	1
$V_{3.18}$	3	36	0	[26]	(9,3,11)	1
$V_{3.19}$	3	38	0	[24]	(7,1,9)	1
$V_{3.19}$	3	38	0	[24]	(8,2,10)	1
$V_{3.20}$	3	38	0	[28]	(7,1,9)	1
$V_{3.20}$	3	38	0	[28]	(8,2,10)	1
$V_{3.20}$	3	38	0	[28]	(9,3,11)	1
$V_{3.21}$	3	38	0	[22]	(8,2,10)	1
$V_{3.22}$	3	40	0		(7,1,9)	1
$V_{3.23}$	3	42	0	[20]	(7,1,9)	1
$V_{3.23}$	3	42	0	[20]	(8,2,10)	1
$V_{3.24}$	3	42	0	[22]	(7,1,9)	1
$V_{3.24}$	3	42	0	[22]	(8,2,10)	1
$V_{3.25}$	3	44	0		(7,1,9)	1
$V_{3.26}$	3	46	0		(7,1,9)	1
$V_{3.28}$	3	48	0		(7,1,9)	1

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
$V_{4.1}$	4	24	1		(12,6,14)	1
$V_{4.2}$	4	28	1		(10,4,12)	1
$V_{4.3}$	4	30	0		(10,3,13)	1
$V_{4.4}$	4	32	0	$[-40]$	(9,2,12)	1
$V_{4.5}$	4	32	0	$[-39]$	(9,2,12)	1
$V_{4.6}$	4	34	0		(10,3,13)	1
$V_{4.6}$	4	34	0		(9,2,12)	1
$V_{4.7}$	4	36	0		(8,1,11)	2
$V_{4.7}$	4	36	0		(9,2,12)	1
$V_{4.8}$	4	38	0		(8,1,11)	1
$V_{4.9}$	4	40	0		(8,1,11)	1

Any smooth Fano threefold not listed in the table does not admit any small toric degenerations, since none of nodal toric Fano threefolds has the proper invariants.

3. Proof

See [1].

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